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Dependent Lifelengths Induced by Dynamic Environments

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In assessing the reliability of systems, it is usual to suppose the component lifelengths are independent. This is inapropriate because the common environment acting on all components induces dependence. Assuming the common environment to be static over time, Lindley and Singpurwalla (1986) proposed a simple model that incorporated dependence. In this dissertation, the more realistic case of dynamic environments is considered and its ramifications explored. The investigation leads to the development of new families of multivariate distributions, one having as a special case the multivariate exponential of Marshall and Olkin (1967).

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DEPENDENT LIFELENGTHS INDUCED BY DYNAMIC ENVIRONMENTS

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by

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ABSTRACT

In assessing the reliability of systems, it is usual to suppose that the component lifelengths are independent. This is inappropriate because the common environment acting on all components induces dependence. Assuming the common environment to be static over time, Lindley and Singpurwalla [1986] proposed a simple model that incorporated dependence. In this dissertation, the more realistic case of dynamic environments is considered and its ramifications explored. The investigation leads to the development of new families of multivariate distributions, one having as a special case the multivariate exponential of Marshall and Olkin [1967].

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CHAPTER I

INTRODUCTION AND OVERVIEW

1.0 Background

Assessing the reliability of series and parallel systems in terms of their component reliabilities is an important practical problem. The component reliabilities are usually assessed via life tests conducted under controlled test-bench environments which are generally static over time. The assessed component reliabilities are then used to estimate system reliability, assuming the independence of component lifelengths and ignoring the effects of the operating environment on them. Testing the entire system or even individual components under actual operational environments is rarely feasible. The operating environment varies from one application to another and is typically harsher or gentler than the test-bench environment. Furthermore, for a particular application, the operating environment may be *dynamic*; that is, it changes over time. For example, the failure of one component could impose an additional

stress on the surviving components and *de facto* constitute a change in the operating environment. When the effect of the operating environment on the system is ignored and the component lifelengths judged independent, the calculated assessments of system reliability fall short of actual observation. Such experiences have cast doubt on the applicability of reliability theory to practical problems.

Multivariate distributions describing dependent lifelengths of biological and engineering systems have been proposed by Freund [1961], Downton [1970], and Marshall and Olkin [1967]. More recently, Lindley and Singpurwalla [1986] - henceforth LS - considered the simple case of a two component parallel redundant system which operates in a static environment different from the test-bench environment. They proposed a family of bivariate distributions, a special case of which is the *bivariate logistic distribution* of Malik and Abraham [1973]. Notwithstanding cumbersome algebra and uninspiring detail (see for example Nayak [1987]), generalization of the LS model to the multivariate case is straightforward.

Our objective here is to expand upon the basic theme of LS in several directions, the most important being a consideration of dynamic operating environments. To facilitate this and to establish some notation and terminology, a brief review of the basic model of LS is necessary.

Suppose that m components in a parallel redundant system are judged to have exponential lifelengths with parameters λ_{1o}, λ_{2o}, ..., λ_{mo}, where the λ's pertain to test-bench environments and are assumed specified. The λ_{io}'s (or more generally the $\lambda_{io}(t)$'s, $t \geq 0$, if the lifelengths are judged to be other than exponential) will be referred to as the *baseline failure rates* of the

components. Assume that the effect of the common static operating environment is to modulate (i.e., to increase or decrease) each λ_{i_o}, $i = 0,1,\ldots,m$, by a common factor η, where η is unknown; suppose that our uncertainty about η is described by a distribution G. The notion that the operating environment - a covariate in the biostatistical literature - influences the baseline failure rate is due to Cox [1972]; ramifications of it are in Kalbfleisch and Prentice [1980]. It is our uncertainty about η that induces dependence among the lifelengths of the components, T_1, T_2, $\ldots$,T_m, which now have a multivariate distribution whose nature is determined by the assumed form for G. For example, when G is taken to be a gamma distribution, T_1, T_2, $\ldots$,T_m have the multivariate logistic distribution of Malik and Abraham [1973].

The behavior of the reliability of series and parallel systems, under the set-up of LS with G a gamma distribution, has been studied by Currit and Singpurwalla [1987]. They described circumstances under which ignoring the influence of the operating environment would lead to optimistic or pessimistic assessments of system reliability. The LS model gives the user an example of the importance of incorporating dependencies in reliability assessment and the consequences of a naïve assumption of independence.

1.1 Extensions of the Lindley-Singpurwalla Theme

Extentions of the basic theme of LS can occur in several directions, the most obvious ones being to assume that the baseline failure rates are no longer constant and/or to choose other forms for G, say the lognormal or the truncated normal. We shall not pursue these lines of development because the underlying analysis is straightforward and routine. A noteworthy extention of the LS theme can be achieved if we allow the operating environment to be dynamic by making η a function of time, for example $\eta(t) = \sum_{k=0}^{s} \beta_k t^k$ or $\eta(t) = \exp\{\sum_{k=0}^{s} \beta_k t^k\}$, where the β_k's are assumed unknown. In the former case, our uncertainties about the β's could be described by a sequence of independent gamma or truncated normal distributions whereas in the latter case we could assume normal distributions. The underlying analysis for either case is cumbersome though straightforward; consequently, we have chosen to merely state the ensuing results in Appendix B. The case $\eta(t) = \exp\{\sum_{k=0}^{s} \beta_k t^k\}$, with dependencies between the β_k's (to ensure a certain type of smoothness), has recently been considered by Gamerman and West [1987] for the analysis of medical survival data.

Instead of assuming that $\eta(t)$ is a polynomial in t, we may assume that $\eta(t)$ is a piecewise constant function of t where the values of the constants are unknown and are described by suitable multivariate distributions which ensure some form of dependence between the constants. A motivation for considering the piecewise constant function is its meaningfulness in

certain applications such as *step-stress testing* (see for example Goel and DeGroot [1979] and Shaked and Singpurwalla [1983]), and also the fact that certain limiting operations on it enables us to describe $\eta(t)$ by suitable stochastic processes; as we shall see, these processes lead to interesting multivariate distributions for the lifelengths.

Recent work has explored the use of stochastic processes to model dynamic environments. Flournoy [1982] has considered time-varying covariate processes as they effect both failure and censoring mechanisms. Cinlar and Õzekici [1987] have used a semi-Markov process to model the environment with a deterministic function relating the environment to the failure rate. We will focus on using stochastic processes to produce useful multivariate distributions.

In what follows, we shall refer to $\eta(t)$, $t \geq 0$, as the *environmental factor function* (EFF). The EFF is a parameter that has little if any direct physical meaning. It is introduced for convenience to capture our opinions about the physical effects of the operating environment on the system. However, the EFF is important, since the nature of the induced component dependencies depend upon the form of the EFF and the manner in which we describe our uncertainty about it.

A criticism of the LS model is that the operating environment is assumed to have the same effect upon all m components of the system. This assumption is manifested by our consideration of a common unknown quantity η or $\eta(t)$ which modulates each baseline failure rate. An obvious extension, therefore, is to assume that the baseline failure rate of component i is modulated by an unknown quantity η_i (or an unknown function $\eta_i(t)$), $i = 0,1,\ldots,m$, and

to describe our uncertainty about the η_i's at any time t by a meaningful multivariate distribution. This idea is pursued in Chapter 2 for the static environment and in succeeding chapters for dynamic environments.

1.2 Outline of Dissertation and Overview of Main Results

In Chapter 2 we extend the LS model by supposing that the operating environment has different effects on the different components. To do this, we propose a multivariate distribution which describes the dependencies between the EFFs for each component and use this to produce a multivariate distribution for the lifelengths of the various components. We also demonstrate that the degree of dependence between the EFFs gets reflected as inequalities for the survival function of parallel redundant systems (see section 2.3). The entire development of Chapter 2 is under the assumption that the operating environment is static.

In Chapters 3 and 4 we consider the case of dynamic environments, with Chapter 3 devoted to the case of the EFF described by a piecewise constant function and Chapter 4 being devoted to the case of the EFF described by a continuous time stochastic process with independent gamma distributed increments. In Chapter 3, the points at which the EFF takes jumps may be specified or random and the sizes of the jumps are assumed to be random and described by gamma distributions. Considered in Chapter 3 (section 3.2) is the case in which the EFF is

described by a "shot-noise" process. A special case of the shot-noise process produces, it appears, a new family of multivariate distributions with exponential marginals. In section 3.3 of Chapter 3, we consider the case of a dynamic environment described by piecewise constant EFFs which has different effects on the different components of the system.

The development of Chapter 4 produces multivariate distributions which appear to be new. The development of the material in section 4.1 provides an alternative derivation of the bivariate exponential distribution of Marshall and Olkin [1967]. In section 4.3 of Chapter 4 we consider the case of a dynamic environment described by a gamma stochastic process for the EFF which has different effects on the different components of the system.

CHAPTER II

DEPENDENT LIFELENGTHS INDUCED BY STATIC ENVIRONMENTS

HAVING DIFFERENT EFFECTS ON EACH COMPONENT

2.0 Introduction

The model proposed by LS assumes that the operating environment is static and that each baseline failure rate λ_{i_o}, $i = 0,1,\ldots,m$, is modulated by a common EFF η, with the uncertainty about η described by a specified distribution function G. A natural extention of the above is to hypothesize that each λ_{i_o} is modulated by a different EFF, say η_i, with our uncertainty about η_i described by a specified distribution function G_i, $i = 0,1,\ldots,m$. It is desirable to assume that the m environmental factor functions are dependent. One motivation for the dependencies among the η_i's is that some factors which constitute the environment - such as temperature - may have an identical effect on all components, whereas the other environmental factors have

different effects on different components. A plausible model for describing such dependencies is given in Section 2.1; it is based on a theme proposed by Cherian [1941] and David and Fix [1961] - see also Johnson and Kotz [1972], pp. 216-220.

2.1 A Model for Dependencies Between Component EFFs

To simplify matters, we shall restrict our attention to the case of a two-component system, so that η_1 and η_2 are the only EFFs that we need consider. Assume that $\eta_i = X_0 + X_i$, where the random quantity X_0 captures the contribution of the common environmental factors on both components, and the random quantity X_i captures the contribution of the other factors on component i, $i = 1,2$. The Cherian-David-Fix setup assumes that X_0, X_1, and X_2 are independent gamma distributed variables with known shape parameters α_i and scale parameters b_i for $i = 0,1,2$, respectively. Clearly, η_1 and η_2 are dependent with a joint density at $\eta_1, \eta_2 > 0$ given by the Cherian-David-Fix - henceforth CDF - bivariate density.

$$f(\eta_1, \eta_2) = e^{-b_1 \eta_1 - b_2 \eta_2} \prod_{i=0}^{2} \left[\Gamma^{-1}(\alpha_i) \right] \prod_{i=0}^{2} b_i^{\alpha_i}$$

$$\cdot \int_{x_0=0}^{\min(\eta_1,\eta_2)} x_0^{\alpha_0-1} (\eta_1 - x_0)^{\alpha_1-1} (\eta_2 - x_0)^{\alpha_2-1} e^{-(b_0-b_1-b_2)\, x_0} \, dx_0 . \tag{2.1}$$

In what follows, and throughout this dissertation, we shall denote the gamma distribution

by writing "$X_i \sim G(\alpha_i, b_i)$," where the gamma pdf is of the form

$$g(x) = \frac{b^\alpha x^{\alpha-1} e^{-bx}}{\Gamma(\alpha)}, \qquad x > 0.$$

2.2 The Bivariate Distribution of Lifelengths

In this section we again focus attention on the two component case, and verify via manipulations identical to those found in LS that under the Cherian-David-Fix distribution for the EFFs η_1 and η_2, the lifelengths T_1 and T_2 of a two-component parallel redundant system have the bivariate survival function

$$\bar{F}(\tau_1, \tau_2) \equiv P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, (\alpha_i, b_i), i=0,1,2 \,] =$$

$$\left[\frac{b_0}{\lambda_{1o}\tau_1 + \lambda_{2o}\tau_2 + b_0}\right]^{\alpha_0} \prod_{i=1}^{2} \left[\frac{b_i}{\lambda_{io}\tau_i + b_i}\right]^{\alpha_i}, \qquad \tau_1, \tau_2 \geq 0. \tag{2.2}$$

The marginal survival function of T_i, $i=1,2$, is

$$\bar{F}_i(\tau_i) \equiv P[\, T_i \geq \tau_i \mid \lambda_{io}, (\alpha_l, b_l), l=0,i \,] =$$

$$\left[\frac{b_0}{\lambda_{io}\tau_i + b_0}\right]^{\alpha_0} \left[\frac{b_i}{\lambda_{io}\tau_i + b_i}\right]^{\alpha_i}, \qquad \tau_i \geq 0. \tag{2.3}$$

Note that the survival function (2.2) is the product of three contributing terms and the marginals (2.3) are each the product of two terms, one representing the common contribution of X_0 and the other the contribution of X_i, $i = 1,2$. The development of (2.2) and (2.3) has been greatly facilitated by the assumption that X_0, X_1, and X_2 are mutually independent. If the scale factors b_0, b_1, and b_2 are equal to the same value, say b, then the marginal survival function (2.3) is a Pareto with parameters $(\alpha_0 + \alpha_i)$ and b/λ_{io}, $i = 1,2$; this result is analogous to that of LS.

A generalization of the above to the multivariate case is straightforward, although it requires the specification of several parameters.

The case of a dynamic environment having different effects on each component is considered in Chapters 3 and 4.

2.3 Inequalities for Survival Functions with Increasing Degrees of Dependence

An examination of the development of the model given by LS and the model (2.2) given above indicates that the nature of dependence between T_1 and T_2 depends upon the nature of the dependence between η_1 and η_2. Note that in the LS case $\eta_1 = \eta_2 = \eta$, so that the dependence between η_1 and η_2 is the strongest possible, being one to one. It is useful to view

dependence between two variables η_1 and η_2 subjectively, in the sense that η_1 dependent on η_2 implies that knowledge about η_1 causes us to alter our assessment of uncertainty about η_2 made prior to receiving knowledge about η_1. If we use the notation $X \stackrel{d}{=} Y$ to indicate that X has the same distribution as Y, a simple way of constructing three pairs of random quantities (η_1, η_2), (η_1, η_2'), and (η_1, η_2'') with differing degrees of dependence is to introduce four mutually independent random quantities X_0, X_1, X_0', and X_1', with $X_0(X_1) \stackrel{d}{=} X_0'(X_1')$, and define $\eta_1 = \eta_2 = X_0 + X_1$, $\eta_2' = X_0 + X_1'$, and $\eta_2'' = X_0' + X_1'$. In order to insure that differences between survival functions are caused by differing degrees of dependence and not different distributional assumptions, we have required that $\eta_2 \stackrel{d}{=} \eta_2' \stackrel{d}{=} \eta_2''$.

If we assume that X_i and $X_i' \sim G(\alpha_i, b_i)$, $i=0,1$, then the pair of EFFs (η_1, η_2) will result in the bivariate survival function given by LS, where

$$\overline{F}_{LS}(\tau_1,\tau_2) = \prod_{i=0}^{1} \left[\frac{b_i}{b_i + \lambda_{1o}\,\tau_1 + \lambda_{2o}\,\tau_2}\right]^{\alpha_i}, \qquad \tau_1, \tau_2 \geq 0\,. \tag{2.5}$$

If we assume that $\alpha_1 = \alpha_2$, the pair (η_1, η_2') will lead to the model (2.2), denoted by $\overline{F}_{CDF}(\tau_1, \tau_2)$, where the subscript CDF indicates that the survival function is based on the Cherian-David-Fix distribution for η_1 and η_2'. Since η_1 and η_2'' are independent, the pair (η_1, η_2'') will lead to the survival function

$$\overline{F}_{IND}(\tau_1,\tau_2) = \prod_{l=1}^{2}\left[\frac{b_0}{b_0 + \lambda_{lo}\,\tau_l}\right]^{\alpha_0} \prod_{m=1}^{2}\left[\frac{b_1}{b_1 + \lambda_{mo}\,\tau_m}\right]^{\alpha_1}, \qquad \tau_1, \tau_2 \geq 0\,, \tag{2.6}$$

the subscript IND denoting the fact that η_1 and η_2" are mutually independent. Let $(X_1, Y_1) \overset{D}{>} (X_2, Y_2)$ denote the fact that the pair (X_1, Y_1) is more (linearly) dependent than the pair (X_2, Y_2). Then $(\eta_1, \eta_2) \overset{D}{>} (\eta_1, \eta_2') \overset{D}{>} (\eta_1, \eta_2'')$. It is easy to verify that the following theorem holds:

<u>Theorem 2.1</u>

$$\overline{F}_{IND}(\tau_1, \tau_2) < \overline{F}_{CDF}(\tau_1, \tau_2) < \overline{F}_{LS}(\tau_1, \tau_2)$$

for all τ_1 , $\tau_2 > 0$, and $\lambda_{1o}, \lambda_{2o} > 0$.

Thus the bivariate survival function of 2-component parallel redundant systems increases with the degree of dependence between the EFFs. This inequality will hold for the multivariate survival functions of all m-component parallel redundant systems constructed in a similar manner, $m \geq 2$. If we set $\tau_1 = \tau_2 = \tau_m$, then we get inequalities for the system survival function of m-component series systems.

CHAPTER III

DEPENDENCE INDUCED BY DYNAMIC ENVIRONMENTS

WITH PIECEWISE CONSTANT ENVIRONMENTAL FACTOR FUNCTIONS

3.0 Introduction

In this chapter, we shall assume that the operating environment is dynamic and that the associated EFF is described by a piecewise constant, right continuous function over known or unknown time intervals $[\, t_j,\, t_{j+1}\,)$, $j = 0,1,\ldots$, where $t_0 \equiv 0$. Specifically, we let

$$\eta(t) = \eta_j\,, \qquad t \in [\, t_j,\, t_{j+1}\,),$$

with the η_j's unknown and assumed to have various distributional forms. Much of our development will be based on the assumption that the dynamic operating environment has the same effect on all m components. We shall also continue to assume that the baseline failure rates of all m components are constants, so that for $t \in [\, t_j,\, t_{j+1}\,)$, the modulated failure rate

of the i-th component, $\lambda_i(t)$, is a piecewise constant $\lambda_{i_o}\ \eta_j$. Given $\lambda_i(t)$ and $\underline{t} \equiv (\ t_1,\ t_2,\ \ldots\ )$, the lifelength T_i of the i-th component has a piecewise exponential distribution. In what follows, we shall make various distributional assumptions on the η_j's and $\underline{t}$, and explore the consequences of such assumptions. In section 3.1, we assume that $\underline{t}$ is known whereas in section 3.2 we assume that it is not. We note that if $\underline{t}$ is known, then it is possible to determine the intervals in which any two times τ_1 and τ_2 lie. For notational convenience, all survival functions are stated for time intervals $[\ 0,\ \tau_1\ )$ and $[\ 0,\ \tau_2\ )$ such that $0 \le t_{n_1} \le \tau_1 < t_{n_1+1} \le t_{n_2} \le \tau_2 < t_{n_2+1}$.

3.1 Describing Each η_j as a Random Sum of Innovations

As a starting scenario, we assume that the operating environment consists of at most s+1 distinct stresses having the same effect on all m components, with the k-th stress contributing a known or unknown factor (innovation) c_k to η_j. We also assume that the effects of the factors c_k, $k = 0,1,\ldots,s$, are additive so that

$$\eta_j = \sum_{k=0}^{s} I_k(t_j)\ c_k\ , \qquad j=0,1,\ldots$$

where $I_k(t_j) = \begin{cases} 1 & \text{if the } k\text{-th stress is present during } [\ t_j,\ t_{j+1}\ )\ , \\ 0 & \text{otherwise.} \end{cases}$

Let $p_{kj} = P(\ I_k(t_j) = 1\ )$ and let $q_{kj} = 1 - p_{kj}$. Assume that $p_{0j} = 1,\ \ \forall\ j$. This

assumption implies that factor c_0 is always present in the operating environment. Note that the factors c_k do not change over time; it is presence or absence of a factor that changes the EFF.

Were we to assume that $I_k(t_j)$ and c_k are known $\forall\ k,j$, the the survival function of T_1 and T_2 would be

$$\overline{F}(\tau_1, \tau_2) \equiv P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, (\, I_k(t_j), c_k\,),\ k=0,1,\ldots,s,\ j=0,1,\ldots]$$

$$= \prod_{k=0}^{s} \Bigg\{ e^{-\, I_k(t_{n_1})\, c_k\, [\lambda_{1o}(\tau_1 - t_{n_1}) + \lambda_{2o}(t_{n_1+1} - t_{n_1})]}\ e^{-\, \lambda_{2o}\, I_k(t_{n_2})\, c_k\, (\tau_2 - t_{n_2})}$$

$$\cdot \prod_{j=0}^{n_1-1} e^{-\, (\lambda_{1o}+\lambda_{2o})\, I_k(t_j)\, c_k\, (t_{j+1} - t_j)} \prod_{j=n_1+1}^{n_2-1} e^{-\, \lambda_{2o}\, I_k(t_j)\, c_k\, (t_{j+1} - t_j)} \Bigg\}.$$

(3.1)

The EFF $\eta_j \equiv \sum_{k=0}^{s} I_k(t_j)\, c_k$ can be evaluated for each time period $[\, t_j, t_{j+1}\,)$ when the indicator variables $I_k(t_j)$ and the factors c_k are assumed unknown by first conditioning on the c_k's known and then averaging over their distributions G_k, or by conditioning on the $I_k(t_j)$'s and then averaging over their distributions.

We assume in this analysis that the innovations are mutually independent; that is,

$I_k(t_j) \perp\!\!\!\perp I_l(t_j) \qquad \forall\ j,\ \ k \neq l$

$I_k(t_j) \perp\!\!\!\perp I_k(t_l) \qquad \forall\ k,\ \ j \neq l$

$c_k \perp\!\!\!\perp c_l \qquad k \neq l$

where "X $\perp\!\!\!\perp$ Y" indicates that X is independent of Y.

We denote the vector of all indicator variables as $\underline{I}(\underline{t})$ and the vector of all factors as $\underline{c}$. We begin by conditioning on the vector $\underline{c}$ and recognize that for fixed i, j, k,

$$E_{I_k(t_j)}[\, e^{-\lambda_{io}\,(t_{j+1}-t_j)\, I_k(t_j)\, c_k}\,] = \begin{cases} e^{-\lambda_{io}\,(t_{j+1}-t_j)\, c_k} & \text{w.p. } p_{kj}, \\ 1 & \text{w.p. } q_{kj}. \end{cases}$$

Since our indicator variables $I_k(t_j)$ are mutually independent, we can average over each term to arrive at

$$\bar{F}(\tau_1,\tau_2 \mid \underline{c}) \equiv P[\, T_1 \ge \tau_1,\, T_2 \ge \tau_2 \mid \lambda_{1o},\lambda_{2o},\underline{c},\underline{t},\, p_{kj}\ j=0,1,\ldots,\ k=0,1,\ldots,s] =$$

$$= \prod_{k=0}^{s} \Bigg\{ [\, p_{kn_1}\, e^{-[\lambda_{1o}(\tau_1 - t_{n_1}) + \lambda_{2o}(t_{n_1+1} - t_{n_1})]\, c_k} + q_{kn_1}\,]$$

$$\cdot [\, p_{kn_2}\, e^{-\lambda_{2o}\,(\tau_2 - t_{n_2})\, c_k} + q_{kn_2}\,]$$

$$\cdot \prod_{j=0}^{n_1-1} [\, p_{kj}\, e^{-(\lambda_{1o}+\lambda_{2o})\,(t_{j+1}-t_j)\, c_k} + q_{kj}\,]$$

$$\cdot \prod_{j=n_1+1}^{n_2-1} [\, p_{kj}\, e^{-\lambda_{2o}\,(t_{j+1}-t_j)\, c_k} + q_{kj}\,] \Bigg\}. \tag{3.2}$$

In order to find the unconditional bivariate survival function, the (n_2+1) terms inside the braces must be evaluated. This would be cumbersome for large n_1 and n_2.

Alternatively, we can condition on the indicator variables and write the conditional bivariate survival function as:

$$\overline{F}(\,\tau_1, \tau_2 \mid \underline{I}(\underline{t})\,) \equiv P[\,T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \underline{I}(\underline{t}), \underline{t}, G_k, k{=}0,1,\ldots\,] =$$

$$\prod_{k=0}^{s} G_k^*[\ I_k(t_{n_1})\,\{\lambda_{1o}(\tau_1 - t_{n_1}) + \lambda_{2o}(t_{n_1+1} - t_{n_1})\} + I_k(t_{n_2})\,\lambda_{2o}(\tau_2 - t_{n_2})$$

$$+\sum_{j=0}^{n_1-1} I_k(t_j)\,(\lambda_{1o}+\lambda_{2o})(t_{j+1}-t_j) + \sum_{j=n_1+1}^{n_2-1} I_k(t_j)\,\lambda_{2o}(t_{j+1}-t_j)\ ], \tag{3.3}$$

where G_k^* is the Laplace-Stieltjes transform of G_k. The bivariate survival function is not factorable for $I_k(t_j)$ using standard nonnegative distributions G_k, so evaluating the unconditional bivariate distribution is not possible except under special circumstances (such as found in section 3.1.1). Thus, we are forced to consider the special cases presented below. In section 3.1.1, we examine two cases that lead to independent η_j's, and in section 3.1.2, we examine two cases that lead to dependence between the η_j's.

3.1.1 Independent η_j's Described as a Random Sum of Innovations

The objective of the first part of this section is to produce a scenario which motivates us to describe the η_j's as having independent gamma distributions. Such a setup serves as a

foundation for considering the gamma process (see section 4.1.1) to describe the EFF.

We begin by assuming as before that the indicator variables $I_k(t_j)$ and the factors c_k are unknown and are mutually independent. The distributions will be stated for time intervals $[\ 0\ ,\ \tau_1\ )$ and $[\ 0\ ,\ \tau_2\ )$ such that $0 \leq t_{n_1} \leq \tau_1 < t_{n_1+1} \leq t_{n_2} \leq \tau_2 < t_{n_2+1}$; thus we are interested in the EFFs $\eta_0,\ \eta_1,\ \ldots,\ \eta_{n_2}$. We assume that $n_2 < s$; this prevents us from using s+1 different values of η to solve for $c_0,\ c_1,\ \ldots\ c_s$. Therefore $P[\ \eta_j \mid \eta_0,\ \eta_1,\ \ldots,\ \eta_{j-1}\ ] = P[\ \eta_j\ ]$ for $j = 1,2,\ldots,n_2$, and we conclude that the EFFs are mutually independent.

We further assume that the number of stresses present during time interval $[\ t_j,\ t_{j+1}\ )$ is known - denoted by N_j - although the identities of these stresses are unknown. If we describe our uncertainty about each factor c_k with an identical gamma distribution with parameters α and b, $\eta_0,\ \eta_1,\ \ldots,\ \eta_{n_2}$ are independent gamma distributed random variables with parameters $N_j\alpha$ and b.

Independence between the η_j's may not reflect the actual physical system. However, this independent model serves as a foundation for our consideration of the gamma process for the EFF.

The development of the multivariate survival function follows familiar lines. Specifically, for the case $m = 2$,

$$\overline{F}(\ \tau_1,\ \tau_2\ ) \equiv P[\ T_1 \geq \tau_1,\ T_2 \geq \tau_2 \mid \lambda_{1o},\ \lambda_{2o},\ \underline{t},\ \alpha,\ b,\ N_j\ \ j{=}0{,}1{,}\ldots{,}n_2\ ] =$$

$$\left[\frac{b}{\lambda_{1o}(\tau_1 - t_{n_1}) + \lambda_{2o}(t_{n_1+1} - t_{n_1}) + b}\right]^{N_{n_1}\alpha} \left[\frac{b}{\lambda_{2o}(\tau_2 - t_{n_2}) + b}\right]^{N_{n_2}\alpha}$$

$$\cdot \prod_{j=0}^{n_1-1}\left[\frac{b}{(\lambda_{1o}+\lambda_{2o})(t_{j+1} - t_j) + b}\right]^{N_j\alpha} \prod_{j=n_1+1}^{n_2-1}\left[\frac{b}{\lambda_{2o}(t_{j+1} - t_j) + b}\right]^{N_j\alpha}. \tag{3.4}$$

The marginal survival function for component i, i=1,2 is:

$\overline{F}_i(T_i) \equiv P[T_i \geq \tau_i \mid \lambda_{io}, \underline{t}, \alpha, b, N_j\ j{=}0{,}1{,}\ldots{,}n_i] =$

$$\left[\frac{b}{\lambda_{io}(\tau_i - t_{n_i}) + b}\right]^{N_{n_i}\alpha} \prod_{j=0}^{n_i-1}\left[\frac{b}{\lambda_{io}(t_{j+1} - t_j) + b}\right]^{N_j\alpha}, \tag{3.5}$$

$$0 \leq t_{n_i} \leq \tau_i < t_{n_i+1}.$$

It is easy to verify by using a transformation of the type undertaken by LS that the marginal distribution function of either component reduces to a *piecewise Pareto distribution*, so that (3.4) may be referred to as a *bivariate piecewise Pareto survival function*. The survival functions (3.4) and (3.5) call for the specification of too many hyperparameters and are thus of limited practical value except under special circumstances. However, as stated before, in introducing the above set-up, we lay a foundation for the development of the use of gamma processes to describe the EFF - such processes lead to interesting results.

Another scenario that leads to independent η_j's assumes that the factors c_k are known $\forall\ k$, but the indicator variables $I_k(t_j)$ are not. We assume as before that the $I_k(t_j)$'s are mutually

independent for $k = 0,1,\ldots,s,\quad j = 0,1,\ldots$ Conditioning on $\underline{c}$ known leads to the survival function (3.2). Furthermore, if we assume that $c_k = c,\ k \geq 0$ and $p_{kj} = p_j,\ k > 0$ (recall that $p_{0j} = 1\ \forall\ j$), then

$$\sum_{k=0}^{s} I_k(t_j)\, c_k = c \sum_{k=0}^{s} I_k(t_j)$$

has a discrete distribution with support c, 2c, …, $(s+1)$c with probabilities $\binom{s}{k} p_j{}^k (1-p_j)^{s-k}$, $k = 0,1,\ldots,s$, respectively.

In view of the above,

$\bar{F}(\tau_1, \tau_2) \equiv P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \underline{t}, c, p_j \;\; j=0,1,\ldots] =$

$$= \left[\sum_{k=0}^{s} \binom{s}{k} p_{n_1}{}^k (1-p_{n_1})^{s-k}\, e^{-(k+1)\,[\lambda_{1o}(\tau_1 - t_{n_1}) + \lambda_{2o}(t_{n_1+1} - t_{n_1})]\, c} \right]$$

$$\cdot \left[\sum_{k=0}^{s} \binom{s}{k} p_{n_2}{}^k (1-p_{n_2})^{s-k}\, e^{-(k+1)\,\lambda_{2o}\,(\tau_2 - t_{n_2})\, c} \right]$$

$$\cdot \prod_{j=0}^{n_1-1} \left[\sum_{k=0}^{s} \binom{s}{k} p_j{}^k (1-p_j)^{s-k}\, e^{-(k+1)\,(\lambda_{1o}+\lambda_{2o})\,(t_{j+1} - t_j)\, c} \right]$$

$$\cdot \prod_{j=n_1+1}^{n_2-1} \left[\sum_{k=0}^{s} \binom{s}{k} p_j{}^k (1-p_j)^{s-k}\, e^{-(k+1)\,\lambda_{2o}\,(t_{j+1} - t_j)\, c} \right]. \qquad (3.6)$$

Alternatively, if we assume that $p_{kj} = p_k$ and that $[\, t_j, t_{j+1}) = \Delta t\ \forall\ j$, then

$$\bar{F}(\tau_1, \tau_2) \equiv P[T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \underline{c}, \Delta t, p_k \;\; k=0,1,\ldots,s] =$$

$$\prod_{k=0}^{s} \Bigg\{ [\, p_k\, e^{-[\lambda_{1o}(\tau_1 - t_{n_1}) + \lambda_{2o}\, \Delta t]\, c_k} + q_k \,]\, [\, p_k\, e^{-\lambda_{2o}\,(\tau_2 - t_{n_2})\, c_k} + q_k \,]$$

$$\cdot\, [\, p_k\, e^{-(\lambda_{1o} + \lambda_{2o})\, \Delta t\, c_k} + q_k \,]^{n_1}\, [\, p_k\, e^{-\lambda_{2o}\, \Delta t\, c_k} + q_{kj} \,]^{n_2 - (n_1 + 1)} \Bigg\} . \tag{3.7}$$

Either set of assumptions reduces the number of parameters that need to be specified but does not significantly reduce the algebraic complexity.

3.1.2 Dependent η_j's Described as a Random Sum of Innovations

The two special cases of the random sum of innovations scenario contained in the previous section assumed that the EFFs $\eta_0, \eta_1, \ldots, \eta_{n_2}$ are mutually independent. A more realistic assumption may be that the η_j's have a (time) dependent structure. We examine two cases below that lead to this.

We begin the first case by assuming that the indicator variables $I_k(t_j)$ are known and have the following special structure:

$$I_k(t_j) = \begin{cases} 0 & 0 \leq j < k \\ 1 & j \geq k \end{cases} \qquad \begin{array}{l} j=0,1,\ldots\;, \\ k=0,1,\ldots s\,. \end{array}$$

When the indicator variables have this structure, the EFF η_j for the time period $[\ t_j,\ t_{j+1}\ )$ is

$$\eta_j = \sum_{k=0}^{j} c_k\ , \qquad\qquad j = 0,1,\ldots\ .$$

We assume that each factor c_k is unknown and that $c_k \perp\!\!\!\perp c_l \quad k \neq l$. We describe our uncertainty about each c_k through a specified distribution G_k. Here the number of such distributions is equal to the number of time intervals. A set-up like this would be relevant under step-stess testing or when the environment is described by a stress that is cumulative, such as doses of radiation. Note that the EFF is autocorrelated in time. It is easy to verify that $\mathrm{cov}[\ \eta(s),\ \eta(t)\ ] = \mathrm{Var}[\ \eta(s)\ ]$ for $0 \leq s \leq t$.

If we suppose that each c_k is distributed as $G(\ \alpha_k,\ b_k\ )$, the following bivariate survival function is obtained.

$$\overline{F}(\ \tau_1,\ \tau_2\ ) \equiv P[\ T_1 \geq \tau_1,\ T_2 \geq \tau_2 \mid \lambda_{1o},\ \lambda_{2o}, \underline{I}(\underline{t}), \underline{t},\ (\ \alpha_j,\ b_j\ )\ \ j=0,1,\ldots,n_2\ ]$$

$$= \prod_{j=0}^{n_1} \left[\frac{b_j}{b_j + \lambda_{1o}(\tau_1 - t_j) + \lambda_{2o}(\tau_2 - t_j)} \right]^{\alpha_j} \prod_{j=n_1+1}^{n_2} \left[\frac{b_j}{b_j + \lambda_{2o}(\tau_2 - t_j)} \right]^{\alpha_j} . \tag{3.8}$$

This special case can be transformed into a shot noise model if we assume that the transition times T_1, T_2, ... can be described by a Poisson process. For details, see section 3.2.

A second possible set-up assumes that each η_j is dependent upon the previous η_{j-1}. Suppose that the probability that any given stress is present at time t_j is dependent upon

whether that stress was present at time t_{j-1}. The stress levels remain constant (although unknown) across all time periods so dependence will arise between successive time periods if the indicator variables form a two-state Markov chain. We assume that $P[\ I_k(t_j) = i \mid I_k(t_0), \ldots, I_k(t_{j-1})\ ] = P[\ I_k(t_j) = i \mid I_k(t_{j-1})\ ]$ $j = 1,2,\ldots,$ $i = 0,1$; and assume the following transition probabilities for each j, $j = 1,2,\ldots$, and for $k=1,2,\ldots,s$:

$$P[\ I_k(t_j) = 0 \mid I_k(t_{j-1}) = 0\ ] = p(0,0)$$

$$P[\ I_k(t_j) = 1 \mid I_k(t_{j-1}) = 0\ ] = p(0,1)$$

$$P[\ I_k(t_j) = 0 \mid I_k(t_{j-1}) = 1\ ] = p(1,0)$$

$$P[\ I_k(t_j) = 1 \mid I_k(t_{j-1}) = 1\ ] = p(1,1)$$

Note that $p(0,0) + p(0,1) = 1$ and $p(1,0) + p(1,1) = 1$. Thus we need only specify two transition probabilities. The one-step transition matrix is given as $\mathbf{P}_I$, below.

$$\mathbf{P}_I \equiv \begin{bmatrix} p(0,0) & p(0,1) \\ p(1,0) & p(1,1) \end{bmatrix} \qquad \text{for } j=1,2,\ldots; \qquad k=0,1,\ldots,s$$

As initial conditions, we assume that the probability of starting in state 0 is equal to the probability $p(0,0)$ and that the probability of starting in state 1 is equal to the probability $p(0,1)$ for $k = 1,2,\ldots,s$. We continue to assume that the factor c_0 is always present; thus $P[\ I_0(t_j) = 1\ ] = 1 \quad \forall\ j \geq 0$.

To solve for an unconditional survival function, we note that for any time period j and any factor c_k,

$$E[\ e^{-\ I_k(t_j)\, c_k\, \lambda_o\, (t_{j+1}-t_j)} \mid I_k(t_{j-1}) = 0\ ] =$$

$$e^{-\ c_k\, \lambda_o\, (t_{j+1}-t_j)} \cdot p(0,1) + 1 \cdot p(0,0)\ , \qquad \text{and}$$

$$E[\, e^{-\, I_k(t_j)\, c_k\, \lambda_o\, (t_{j+1}-t_j)} \mid I_k(t_{j-1}) = 1\,] =$$

$$e^{-\, c_k\, \lambda_o\, (t_{j+1}-t_j)} \cdot p(1,1) + 1 \cdot p(1,0)\,.$$

Thus for any $j > 1$,

$$\begin{bmatrix} E[\, e^{-\, I_k(t_j)\, c_k\, \lambda_o\, (t_{j+1}-t_j)} \mid I_k(t_{j-1}) = 0\,] \\ E[\, e^{-\, I_k(t_j)\, c_k\, \lambda_o\, (t_{j+1}-t_j)} \mid I_k(t_{j-1}) = 1\,] \end{bmatrix}$$

$$= \quad \mathbf{P_I} \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-\, c_k\, \lambda_o\, (t_{j+1}-t_j)} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Because $\left[\; P(\, I_k(t_0)\,) = 0\,,\, P(\, I_k(t_0)\,) = 1 \;\right] = [\,1\,,\,0\,] \cdot \mathbf{P_I}$, we can evaluate the survival function as a product of matrices $\mathbf{P}_{kj}$ as defined below. Thus

$$\overline{F}(\,\tau_1,\, \tau_2\,) \equiv P[\, T_1 \geq \tau_1,\, T_2 \geq \tau_2 \mid \lambda_{1o},\, \lambda_{2o},\, \underline{c},\, \underline{t},\, p(0,1),\, p(1,0)\,] =$$

$$e^{-\, c_0\, [\, \lambda_{1o}\tau_1 \,+\, \lambda_{2o}\tau_2\,]} \prod_{k=1}^{s} \left\{ [\,1,0\,] \left[\prod_{j=0}^{n_2} \mathbf{P}_{kj} \right] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \tag{3.9}$$

where

$$\mathbf{P}_{kj} \equiv \quad \mathbf{P_I} \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-c_k(\lambda_{1o}+\lambda_{2o})(t_{j+1}-t_j)} \end{bmatrix} \quad \begin{array}{l} j=0,1,\ldots,n_1-1 \\ k=1,\ldots,s \end{array}$$

$$\mathbf{P}_{kj} \equiv \qquad \mathbf{P}_{\mathrm{I}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-c_k\lambda_{1o}(\tau_1 - t_{n_1}) + \lambda_{2o}(t_{n_1+1} - t_{n_1})} \end{bmatrix} \quad \begin{array}{l} j=n_1 \\ k=1,\ldots,s \end{array}$$

$$\mathbf{P}_{kj} \equiv \qquad \mathbf{P}_{\mathrm{I}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-c_k(\lambda_{2o})(t_{j+1} - t_j)} \end{bmatrix} \quad \begin{array}{l} j=n_1+1,\ldots,n_2-1 \\ k=1,\ldots,s \end{array}$$

$$\mathbf{P}_{kj} \equiv \qquad \mathbf{P}_{\mathrm{I}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-c_k(\lambda_{2o})(\tau_2 - t_{n_2})} \end{bmatrix} \quad \begin{array}{l} j=n_2 \\ k=1,\ldots,s \end{array}$$

If the factors c_k $k = 0,1,\ldots,s$ are also unknown, we can evaluate the unconditional survival function by evaluating the product of the matrices. This will be cumbersome for large n_1, n_2, and s.

$$\overline{F}(\tau_1, \tau_2) \equiv P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \underline{t}, G_k, {\scriptstyle k=0,1,\ldots,s}, p(0,1), p(1,0)\,] =$$

$$G_0^*[\, \lambda_{1o}\tau_1 + \lambda_{2o}\tau_2\,] \prod_{k=1}^{s} \left\{ E_{c_k}\left[\; [\,1,0\,] \left[\prod_{j=0}^{n} \mathbf{P}_{kj} \right] \left[\begin{matrix} 1 \\ 1 \end{matrix} \right] \; \right] \right\} \tag{3.10}$$

where $G_0^*[s]$ is the LST of the distribution of c_0.

For example, suppose $0 = t_0 < t_1 \leq \tau_1 < t_2 \leq \tau_2 < t_3$, where $c_k \sim G(\alpha_k, b_k)$:

$$\bar{F}(\tau_1, \tau_2) = \left[\frac{b_0}{b_0 + \lambda_{1o}\tau_1 + \lambda_{2o}\tau_2} \right]^{\alpha_0}$$

$$\cdot \prod_{k=1}^{3} \left\{ p(0,0)^3 + p(0,1)\, p(1,0)\, p(0,0) \left[\frac{b_k}{b_k + (\lambda_{1o} + \lambda_{2o})(t_1 - t_0)} \right]^{\alpha_k} \right.$$

$$+ p(0,0)\, p(0,1)\, p(1,0) \left[\frac{b_k}{b_k + \lambda_{1o}(\tau_1 - t_1) + \lambda_{2o}(t_2 - t_1)} \right]^{\alpha_k}$$

$$+ p(0,0)^2\, p(0,1) \left[\frac{b_k}{b_k + \lambda_{2o}(\tau_2 - t_2)} \right]^{\alpha_k}$$

$$+ p(0,1)\, p(1,1)\, p(1,0) \left[\frac{b_k}{b_k + \lambda_{1o}(\tau_1 - t_0) + \lambda_{2o}(t_2 - t_0)} \right]^{\alpha_k}$$

$$+ p(0,0)\, p(0,1)\, p(1,1) \left[\frac{b_k}{b_k + \lambda_{2o}(\tau_1 - t_1) + \lambda_{2o}(\tau_2 - t_1)} \right]^{\alpha_k}$$

$$+ p(0,1)\, p(1,0)\, p(0,1) \left[\frac{b_k}{b_k + (\lambda_{1o} + \lambda_{2o})(t_1 - t_0) + \lambda_{2o}(\tau_2 - t_2)} \right]^{\alpha_k}$$

$$\left. + p(1,0)\, p(1,1)^2 \left[\frac{b_k}{b_k + \lambda_{1o}(\tau_1 - t_0) + \lambda_{2o}(\tau_2 - t_0)} \right]^{\alpha_k} \right\}. \qquad (3.11)$$

3.2 Modeling the EFF as a Shot-Noise Process

Suppose that the operating environment consists of a series of "shots" (or "shocks") of stress whose magnitudes X_k, $k = 0,1,\ldots$, are unknown. Suppose that the shocks occur over time per the postulates of a nonhomogeneous Poisson process with a specified rate m(t), $t \geq 0$, and that whenever a shock of magnitude X occurs at epoch s, its contribution to the EFF at time s + t is $X \cdot h(t)$, where $h(\cdot)$ is a positive and non-increasing function of $t \geq 0$; $h(\cdot)$ is called the *attenuation function*. Suppose that $T_{(0)}$, $T_{(1)}$, ..., are the successive epochs at which shocks of magnitude X_0, X_1, ..., respectively, occur; then the EFF at time t is

$$\eta(t) = \sum_{k=0}^{\infty} X_k \, h(\, t - T_{(k)}\,) \, ,$$

where $h(u) = 0$, for all $u < 0$. The process $\{ \eta(t), t \geq 0 \}$ is called a *shot-noise process*; see for example, Cox and Isham [1980] p. 135.

If the baseline failure rate of each component is constant, then modeling the EFF as a shot-noise process is equivalent to modeling the failure rate of each component as a shot-noise process.

In Figure 3.1 below, we show the sample path of a shot-noise process for $\eta(t)$; to ensure that the baseline failure rates are not annihilated, we have assumed that observation on the environmental process starts at $T_0 \equiv 0$, at which time the first shock of magnitude X_0 necessarily occurs.

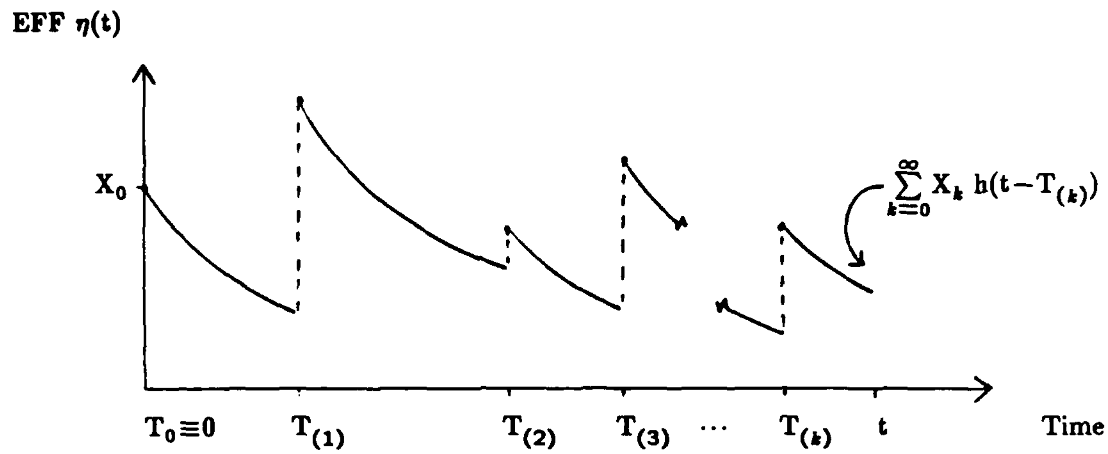

Figure 3.1 A Sample Path of a Shot-Noise Process

The use of a shot-noise process for modeling life-lengths is made possible by the following two assumptions:

A1. $X_k \perp\!\!\!\perp T_{(k)}$, $\forall\ k$; this assumption states that the magnitude of each shot is independent of when the shot occurs.

A2. The X_k's are independent and identically distributed as a random variable, say X.

Suppose that X has a distribution G, and that $G_X^*(s)$ is the Laplace-Stieltjes transform of G. Then the following theorem specifies the survival function, in the operating environment, of a single component system with a constant baseline failure rate λ_o.

Theorem 3.1

Let $M(t) = \int_0^t m(u)\, du$, $H(t) = \int_0^t h(u)\, du$, and $\eta(t) = \sum_{k=0}^{\infty} X_k\ h(\ t-T_{(k)}\ )$, where

$T_{(0)} = 0$. Then under A1, A2, the lifelength T of a single component system with a baseline failure rate λ_o has a survival function for $\tau \geq 0$

$$\overline{F}(\,\tau \mid \lambda_o\,) = G_X^*[\,\lambda_o\, H(\tau)\,]\, e^{-\,M(\tau)}\, e^{\int_0^\tau G_X^*[\,\lambda_o\, H(u)\,]\, m(\tau - u)\, du} .$$

The proof of theorem 3.1 is facilitated via the following [possibly well known (cf. Lemoine and Wenocur [1986]) but important] lemma. A proof is provided to lay the necessary groundwork for the proof of another lemma (Lemma 3.4).

<u>Lemma 3.2</u>

Suppose that $T_{(1)} < T_{(2)} < \cdots < T_{(n)}$ are the epochs of n occurrences in a nonhomogeneous Poisson process with a rate m(t), t≥0, in the time interval $[\,0,\, \tau\,)$. Then T_i, $i = 1,\ldots,n$, the *unordered* epochs, are independent and identically distributed as a random variable, say U, whose density at u, $0 \leq u < \tau$, is $\dfrac{m(u)}{M(\tau)}$, where $M(\tau) = \int_0^\tau m(u)\, du$.

<u>Proof of Lemma 3.2:</u>

If we define a new time variable, say u, where $U(t) = \int_0^t m(u)\, du$, then on the new time scale, the epochs $U_{(j)} = \int_0^{T_{(j)}} m(u)\, du$, with $U_{(1)} \leq U_{(2)} \leq \cdots$, are described by a homogeneous Poisson process with a rate 1 (see Cox and Isham [1980], p. 48). Therefore, given the number of shocks n, the joint distribution of the ordered epochs $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)}$ is the same

as the order statistics in a sample of size n from a uniform distribution on $[\,0,\ U(\tau)\,)$, from which it follows that the unordered epochs U_i , $i = 1,2,\ldots,n$, are uniformly and independently distributed on $[\,0,\ U(\tau)\,)$. But $U(\tau) = M(\tau)$, and so $P[\,U_i \le u\,] = P[\int_0^{T_i} m(u)\,du \le u\,] = P[\,M(T_i) \le u\,] = P[\,T_i \le M^{-1}(u)\,] = \frac{u}{M(\tau)}$, assuming that M^{-1} exists. It now follows that since $P[\,T_i \le M^{-1}(u)\,] = \dfrac{M\,M^{-1}(u)}{M(\tau)}$, $P[\,T_i \le u\,] = \dfrac{M(u)}{M(\tau)}$, and thus the probability density of T_i at any u is $\dfrac{m(u)}{M(\tau)}$. □

Proof of Theorem 3.1:

Let $\Lambda(\tau) = \int_0^{\tau} \lambda_o\,\eta(u)\,du$ and suppose that we condition on $n \ge 1$ shocks occurring in $[\,0,\ \tau\,)$ at times $\underline{T} \equiv (\,T_{(0)} < T_{(1)} < \cdots < T_{(n)}\,)$ with each shot having an (unknown) magnitude X; recall that by assumption a shot must necessarily occur at $T_{(0)} \equiv 0$. Then

$$\overline{F}(\,\tau \mid \lambda_o,\ n,\ \underline{T},\ X\,) = e^{-\left[\lambda_o\, X\, H(\tau) \,+\, \lambda_o \sum_{k=1}^{n} X\, H(\tau - T_{(k)})\right]},$$

which because of Lemma 3.1 can also be written as

$$\overline{F}(\tau \mid \cdot\,) = \left[e^{-\,\lambda_o\, X\, H(\tau)}\right]\left[e^{-\,\lambda_o\, X\, H(\tau-U)}\right]^{n}.$$

Note that the above step would not have been possible if A2 did not hold. If we remove the conditioning on X, then

$$\overline{F}(\,\tau \mid \lambda_o,\ n,\ \underline{T}\,) = G_X^*[\,\lambda_o\, H(\tau)\,]\left[\,G_X^*[\,\lambda_o\, H(\tau-U)\,]\,\right]^{n},$$

and unconditioning on U, we have

$$\overline{F}(\,\tau \mid \lambda_o,\ n\,) = G_X^*[\,\lambda_o\, H(\tau)\,]\left[\,\int_0^{\tau} G_X^*[\,\lambda_o\, H(\tau-u)\,]\,\frac{m(u)}{M(\tau)}\,du\,\right]^{n}.$$

Our final step is to uncondition on $n \geq 0$ - which has probability mass function $e^{-M(\tau)} \frac{[M(\tau)]^n}{n!}$ - and simplify the ensuing expressions to get the desired result. □

The function h(u) is normally chosen to be a decreasing function of time (see for example Lemoine and Wenocur [1985,1986]). We have adapted the shot-noise model to accomodate a constant stress between epochs, consistent with the postulates of this chapter. For a specific example, suppose that $X \sim G(\alpha, b)$, $h(u) = 1$, and $m(u) = m$, for all $u \geq 0$. Then, for $\alpha \neq 1$, it can be seen that for $\tau \geq 0$,

$$\overline{F}(\tau) = \left[\frac{b}{b + \lambda_o \tau}\right]^{\alpha} e^{\left\{ \frac{mb}{\lambda_o(\alpha - 1)}\left[1 - \left[\frac{b}{b + \lambda_o \tau}\right]^{\alpha - 1} \right] - m\tau \right\}}, \tag{3.12}$$

and for $\alpha = 1$,

$$\overline{F}(\tau) = \left[\frac{b + \lambda_o \tau}{b}\right]^{\frac{mb}{\lambda_o} - 1} e^{-m\tau} . \tag{3.13}$$

Furthermore, if in the above expression we set $m = \frac{\lambda_o}{b}$, then

$$\overline{F}(\tau \mid \lambda_o, b) = e^{-m\tau} , \tag{3.14}$$

implying that the lack of memory property of the component lifelength in the test environment is preserved in an operating environment described by a shot-noise process, if the shot

magnitudes are exponential with a scale parameter b and the shot arrival rate is the ratio of the baseline failure rate to b. This result is intriguing and invites some form of interpretation; this is given below.

3.2.1 Behavior of the Failure Rate Function

We can easily verify that the failure rate of (3.13) is for $\tau \geq 0$

$$r(\tau) = \frac{\lambda_o(1 + m\tau)}{b + \lambda_o\tau} = \frac{\frac{\lambda_o}{b}(1 + m\tau)}{1 + \frac{\lambda_o}{b}\tau} \; ; \tag{3.15}$$

thus $r(0) = \frac{\lambda_o}{b}$ and $\lim_{\tau\to\infty} r(\tau) = m$. Upon examining the derivative of $r(\tau)$,

$$\text{namely } \frac{d}{d\tau} r(\tau) = \frac{(m\frac{b}{\lambda_o} - 1)}{(\frac{b}{\lambda_o} + \tau)^2}, \quad \tau > 0, \tag{3.16}$$

we note that when $\frac{\lambda_o}{b} < (>)\ m$, then $\frac{d}{d\tau} r(\tau) > (<)\ 0$ for $\tau \in [\,0,\infty\,)$, thus $r(\tau)$ increases (decreases) from $\frac{\lambda_o}{b}$ to m; when $\frac{\lambda_o}{b} = m$, $\frac{d}{d\tau} r(\tau) = 0$, implying that $r(\tau)$ is constant at m. Thus the behavior of the failure rate function depends on the relationship between the parameters $\frac{\lambda_o}{b}$ and m; see Figure 3.2.

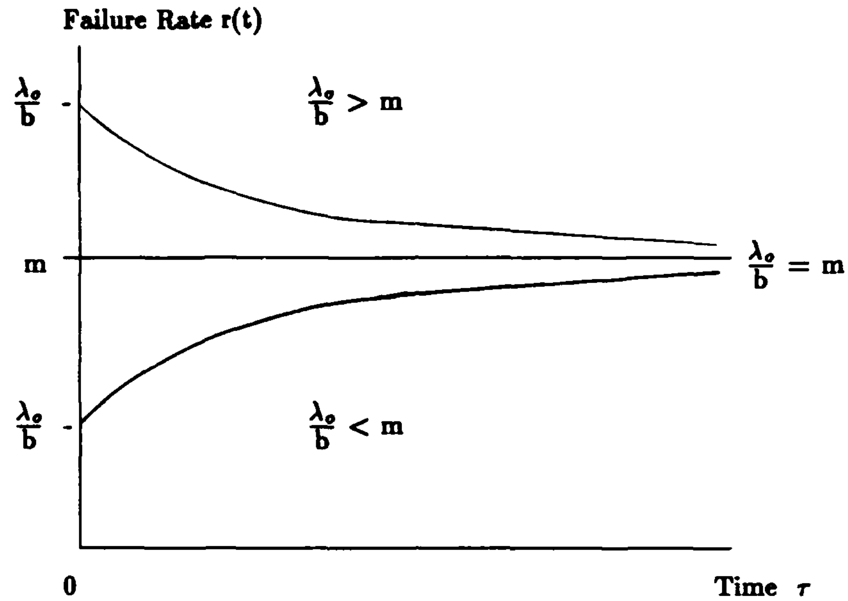

Figure 3.2. Behavior of the Single Component Failure Rate Function with Varying $\frac{\lambda_o}{b}$.

An intuitive explanation of the monotone increasing, decreasing, or constant behavior of the failure rate function can be given if we interpret the failure rate function subjectively - as it should be (cf. Singpurwalla [1988]). When the component with a baseline failure rate of λ_o is placed into the operating environment, it receives a shot of unknown magnitude X_0; this modulates its failure rate to $\lambda_o X_0$. Since X_0 is unknown, it is reasonable for us to estimate it at its expected value $\frac{1}{b}$, making $r(0) = \frac{\lambda_o}{b}$. As time passes, we learn more about the survival (or failure) of the item and this plus our expectations about the shot arrival process causes us to revise (upwards or downwards) our assessment of the failure rate from its current value. For example, if the operating environment is such that there are no shocks on (0^+, ∞)

[subsequent to the first shot of magnitude X_0 at 0], then the expected time to failure of the item will be $\frac{b}{\lambda_o}$. However, if shots do occur (at a rate m), then the expected time between the occurrence of the shots will be $\frac{1}{m}$. At the occurrence of each shot, the propensity of the item to fail will increase and so will our assessment of the failure rate. If $\frac{1}{m} = \frac{b}{\lambda_o}$, equivalent to $m = \frac{\lambda_o}{b}$, we will expect the item to fail due to the effect of $\lambda_o X_0$ at about the same time as the occurrence of the first shock. Under these circumstances, the failure rate of the item (as viewed by us) will be constant at $\frac{\lambda_o}{b}$ because (in our opinion) the dominant failure causing mechanism will be due to the λ_o and the X_0 combination and not the damage due to the shocks.

If $\frac{\lambda_o}{b} < m$, or $\frac{b}{\lambda_o} > \frac{1}{m}$, we will expect the first shock to arrive prior to failure of the item due to the effect of $\lambda_o X_0$; given that the item has not failed at any time τ, our assessment of the failure rate will increase since we expect that the dominant failure mechanism is governed by the shot noise process. It is helpful to conceptualize the situation as being akin to that of competing risks wherein the two competing mechanisms of failure are the shot process over $(0^+, \infty)$, and the modulated failure process determined by $\lambda_o X_0$.

If $\frac{\lambda_o}{b} > m$, or $\frac{b}{\lambda_o} < \frac{1}{m}$, we would expect the first shock to occur after the item has failed due to the effect of $\lambda_o X_0$. Given that the item survives to some time τ, our judgement is that the estimate of the starting value of the failure rate, $\frac{\lambda_o}{b}$, was too high and therefore the failure rate should be lowered. Note that (in our view) the dominant failure mechanism here is not the shot noise process. The arguement used here is analogous to that used by Barlow [1985] to explain why mixtures of exponential distributions can lead to distributions with decreasing

failure rates.

Finally, under both cases $\frac{\lambda_o}{b} > m$ and $\frac{\lambda_o}{b} < m$, the question arises as to why the failure rate will asymptote at m. We can explain this by noting that in a shot noise process with additive damage (like the one considered here), the time to failure of a component can be viewed as the time at which the cumulative damage exceeds a threshold. Given that the time between the shots has an exponential distribution and that the item has not failed at some large τ, its failure rate at τ will behave like the failure rate of a multiple standby system comprising of identical components with exponential lifelengths. The lifelength of such a system has a gamma distribution whose failure rate asymptotes at the failure rate of each component - in our case, the quantity m.

3.2.2 The Bivariate Case

The bivariate survival function of a two-component parallel redundent system with constant baseline failure rates λ_{1_o} and λ_{2_o} is provided by Theorem 3.3 below. Its proof is facilitated by Lemma 3.4, which is analogous to Lemma 3.2.

Theorem 3.3

If M(t), H(t), and $\eta(t)$ are as defined in Theorem 3.1, then under A1 and A2, the lifelengths T_1

and T_2 of a two-component parallel redundent system have the bivariate survival function for $0 \le \tau_1 \le \tau_2$

$$\overline{F}(\tau_1,\tau_2 \mid \lambda_{1o}, \lambda_{2o}) = G^*_X[\lambda_{1o} H(\tau_1) + \lambda_{2o} H(\tau_2)]$$

$$\cdot\ e^{\int_0^{\tau_1} G^*_X [\lambda_{1o} H(\tau_1 - u_1) + \lambda_{2o} H(\tau_2 - u_1)]\ m(u_1)\, du_1}$$

$$\cdot\ e^{\int_{\tau_1}^{\tau_2} G^*_X [\lambda_{2o} H(\tau_2 - u_2)]\ m(u_2)\, du_2} \cdot e^{-M(\tau_2)} . \tag{3.17}$$

Lemma 3.4

Suppose that $T_{(n_1+1)} < T_{(n_1+2)} < \cdots < T_{(n_2)}$ are the epochs of occurrences in a nonhomogeneous Poisson process with rate $m(t)$, $t \ge 0$, in the time interval $[\tau_1, \tau_2)$. Then T_i, $i = (n_1+1), \ldots, (n_2)$, the *unordered* epochs, are independent and identically distributed as a random variable, say U, whose density at u, $\tau_1 \le u < \tau_2$ is $\dfrac{m(u)}{M(\tau_2) - M(\tau_1)}$.

Proof of Lemma 3.4:

Following the proof of Lemma 3.2, if we define a new time variable $U(t)$, then given the number of epochs in $[\tau_1, \tau_2)$, the joint distribution of the ordered epochs $U_{(n_1+1)} < U_{(n_1+2)} < \cdots < U_{(n_2)}$ is the same as that of order statistics in a sample of size $n_2 - n_1$ from a uniform distribution on $[U(\tau_1), U(\tau_2))$, where $U(T_i) = M(T_i)$. The remainder of the arguement is the same as that of Lemma 3.2. □

Proof of Theorem 3.3:

This proof parallels the proof of Theorem 3.1 with the additional proviso that for Poisson processes, the number of events in [τ_1, τ_2) is independent of the number of events in [0, τ_1) - a consequence of the independent increments property. □

Remark 3.5 The marginal distributions of T_1 and T_2 follow from the statement of Theorem 3.3, by setting either τ_1 or τ_2 equal to 0. Specifically, if we set $\tau_1 = 0$, then, for $\tau_2 \geq 0$,

$$\overline{F}(\tau_2 \mid \lambda_{2o}) = G_X^*[\lambda_{2o} H(\tau_2)]$$

$$\cdot \; e^{\int_0^{\tau_2} G_X^* [\lambda_{2o} H(\tau_2-u_2)] \; m(u_2) \, du_2} \cdot e^{- M(\tau_2)} . \tag{3.18}$$

This is in agreement with the statement of Theorem 3.1.

By way of specifics, suppose that $X \sim G(1, b)$, $h(u) = 1$, and $m(u) = m$, for all $u \geq 0$. Then Theorem 3.3 leads us to what appears to be a *new* family of bivariate distributions with exponential marginals. Specifically, for $0 \leq \tau_1 \leq \tau_2$,

$\overline{F}(\tau_1,\tau_2 \mid \lambda_{1o}, \lambda_{2o}, b, m) =$

$$\left[\frac{b}{b + \lambda_{1o}\tau_1 + \lambda_{2o}\tau_2}\right] \left[\frac{b + \lambda_{2o} (\tau_2-\tau_1)}{b + \lambda_{1o}\tau_1 + \lambda_{2o}\tau_2}\right]^{-\frac{mb}{(\lambda_{1o}+\lambda_{2o})}}$$

$$\cdot \left[\frac{b}{b + \lambda_{2o}(\tau_2-\tau_1)}\right]^{-\frac{mb}{\lambda_{2o}}} e^{- m \tau_2} , \tag{3.19}$$

and

$$\overline{F}_2(\ \tau_2 \mid \lambda_{2o}, b, m\ ) = \left[\frac{b + \lambda_{2o}\,\tau_2}{b}\right]^{\frac{mb}{\lambda_{2o}} - 1} e^{-\,m\tau_2}, \qquad \tau_2 \geq 0\ . \tag{3.20}$$

If we now assume that $\lambda_{1o} = \lambda_{2o} = \lambda$ and that $m = \frac{\lambda}{b}$, then, for $0 \leq \tau_1 \leq \tau_2$,

$$\overline{F}(\ \tau_1, \tau_2 \mid m\ ) = \sqrt{\frac{1 - m\tau_1 + m\tau_2}{1 + m\tau_1 + m\tau_2}}\ e^{-\,m\,\tau_2}\ , \tag{3.21}$$

and for $\tau_2 \geq 0$,

$$\overline{F}_2(\tau_2) = e^{-\,m\tau_2}\ , \tag{3.22}$$

an exponential distribution. The results for $\tau_2 \leq \tau_1$ and $\overline{F}_1(\tau_1)$ are symmetric.

The results of this section can be generalized to multi-component systems and also to the case in which the occurrence of a shot at T_j does not necessarily lead to a damage X_j , but associated with each occurrence of a shot is a probability, say p, of a damage X_j . When this is the case, the underlying shot occurrence process is a thinned Poisson process and the development above follows *mutatis mutandis.*

3.3 Piecewise Continuous EFFs Having Different Effects on Each Component

In this section, we consider an explicit dependent structure between the EFFs for each component i, i=1,2,...,m. We assume that each EFF $\eta_i(t)$ is piecewise continuous, sharing the same transition times $\underline{t}$ = (t_1, t_2, ...). The Cherian-David-Fix (CDF) multivariate model is used to describe the dependence between components during each period [t_j, t_{j+1}). For simplicity, we consider only the two-component parallel structure. Let X_{0j} describe the contribution of the environmental stress common to both components during the time interval [t_j, t_{j+1}), and let X_{ij} describe the contribution of the other stresses to component i, i=1,2, during that interval. We assume that the X_{ij}'s are independently distributed as G(α_{ij}, b_{ij}). Thus

$$X_{ij} \perp\!\!\!\perp X_{il} \qquad \forall\ i, \quad j \neq l, \text{ and}$$

$$X_{ij} \perp\!\!\!\perp X_{lj} \qquad \forall\ j, \quad i \neq l\,.$$

3.3.1 Component EFFs Modeled as Time-Independent Gamma Distributed Variables

If we assume that $\eta_{1j} = X_{0j} + X_{1j}$ and $\eta_{2j} = X_{0j} + X_{2j}$, the bivariate density of (η_{1j}, η_{2j}) for each $j \geq 0$ is clearly the CDF density given by (2.1). We note that the component EFFs within each time interval are dependent; that is, η_{1j} and η_{2j} are dependent for each j. However, the EFFs between different time intervals are independent; that is, $\eta_{ij} \perp\!\!\!\perp \eta_{lm}$ $j \neq m$ for $i,l=1,2$. This is analogous to section 2.1 within each time period $[t_j, t_{j+1})$.

The survival function consists of the product of contributions of the form of (2.2) for each interval $[t_j, t_{j+1})$. If $0 \leq \tau_1 \leq \tau_2 < t_1$, the survival function is identical to (2.2).

$$\bar{F}(\tau_1,\tau_2) \equiv P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \underline{t}, (\alpha_{ij}, b_{ij})\ i=0,1,2,\ j=0,1,\ldots]$$

$$= \left[\frac{b_{0n_1}}{\lambda_{1o}(\tau_1 - t_{n_1}) + \lambda_{2o}(t_{n_1+1} - t_{n_1}) + b_{0n_1}}\right]^{\alpha_{0n_1}}$$

$$\cdot \left[\frac{b_{1n_1}}{\lambda_{1o}(\tau_1 - t_{n_1}) + b_{1n_1}}\right]^{\alpha_{1n_1}} \left[\frac{b_{2n_2}}{\lambda_{2o}(\tau_2 - t_{n_2}) + b_{2n_2}}\right]^{\alpha_{2n_2}}$$

$$\cdot \left\{ \prod_{j=0}^{n_1-1} \left[\frac{b_{0j}}{(\lambda_{1o}+\lambda_{2o})\,(t_{j+1}-t_j)+b_{0j}} \right]^{\alpha_{0j}} \prod_{j=0}^{n_1-1} \left[\frac{b_{1j}}{\lambda_{1o}(t_{j+1}-t_j)+b_{1j}} \right]^{\alpha_{1j}} \right.$$

$$\left. \cdot \prod_{j=0}^{n_2-1} \left[\frac{b_{2j}}{\lambda_{2o}(t_{j+1}-t_j)+b_{2j}} \right]^{\alpha_{2j}} \prod_{j=n_1+1}^{n_2-1} \left[\frac{b_{0j}}{\lambda_{2o}\,(t_{j+1}-t_j)+b_{0j}} \right]^{\alpha_{0j}} \right\}. \tag{3.23}$$

The marginals are the product of terms similar to (2.3) for each interval $[\, t_j,\, t_{j+1}\,)$, for $i = 1,2$:

$$\overline{F}_i(\tau_i) \equiv P[\, T_i \geq \tau_i \mid \lambda_{io}, \underline{t}, (\, \alpha_{lj}, b_{lj}\,)\;\; l=0,i \;\; j=0,1,\ldots\,] =$$

$$\left[\frac{b_{0n_i}}{\lambda_{io}(\tau_i - t_{n_i})+b_{0n_i}} \right]^{\alpha_{0n_i}} \left[\frac{b_{in_i}}{\lambda_{io}(\tau_i - t_{n_i})+b_{in_i}} \right]^{\alpha_{in_i}}$$

$$\cdot \prod_{j=0}^{n_i-1} \left\{ \left[\frac{b_{0j}}{\lambda_{io}\,(t_{j+1}-t_j)+b_{0j}} \right]^{\alpha_{0j}} \left[\frac{b_{ij}}{\lambda_{io}(t_{j+1}-t_j)+b_{ij}} \right]^{\alpha_{ij}} \right\}, \tag{3.24}$$

$$0 \leq t_{n_i} \leq \tau_i < t_{n_i+1} \,.$$

If the scale factors for all component distributions are equal, that is, $b_{0j} = b_{1j} = b_{2j} = b_j$ for $j = 0,1,\ldots$, then the marginal distributions (3.24) are piecewise Pareto and (3.23) is bivariate piecewise Pareto.

3.3.2 Component EFFs Modeled as Time-Dependent Gamma Distributed Variables

If we assume that $\eta_{ij} = \eta_{i,j-1} + (X_{0j} + X_{1j})$, $i = 1,2$, $j = 1,2,\ldots$, with $\eta_{i0} = X_{00} + X_{i0}$, where each $X_{ij} \sim G(\alpha_{ij}, b_{ij})$ and all X_{ij}'s are mutually independent, then

$$\eta_{ij} = \sum_{k=0}^{j} (X_{0k} + X_{ik}) , \quad i=1,2 ; \ j=0,1,\ldots \ .$$

We recognize that this forms a time-dependent model similar to that found in section 3.1.2, with (η_{ij}, η_{il}) i = 1,2, autocorrelated when $j \neq l$ and (η_{1j}, η_{2j}) dependent for each $j \geq 0$. The bivariate distribution is similar to (3.8), with separate terms for X_{0j}, X_{1j} and X_{2j}.

$$\overline{F}(\tau_1,\tau_2) \equiv P[T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \underline{t}, (\alpha_{ij}, b_{ij})\ i=0,1,2,\ j=0,1,\ldots]$$

$$= \prod_{j=0}^{n_1} \left[\frac{b_{0j}}{\lambda_{1o}(\tau_1-t_j)+\lambda_{2o}(\tau_2-t_j)+b_{0j}} \right]^{\alpha_{0j}} \prod_{j=n_1+1}^{n_2} \left[\frac{b_{0j}}{\lambda_{2o}\ (\tau_2-t_j)+b_{0j}} \right]^{\alpha_{0j}}$$

$$\cdot \prod_{j=0}^{n_1} \left[\frac{b_{1j}}{\lambda_{1o}(\tau_1-t_j)+b_{1j}} \right]^{\alpha_{1j}} \prod_{j=0}^{n_2} \left[\frac{b_{2j}}{\lambda_{2o}(\tau_2-t_j)+b_{2j}} \right]^{\alpha_{2j}}. \qquad (3.25)$$

3.4 Inequalities for Piecewise Continuous Survival Functions with Increasing Degrees of Dependence

We can examine the effect of increasing degrees of dependence between components 1 and 2 by constructing three pairs of random quantities (η_{1j}, η_{2j}), (η_{1j}, η_{2j}'), and (η_{1j}, η_{2j}") for each j = 0,1,... in the same manner as in section 2.3. We define mutually independent random variables X_{0j}, X_{0j}', X_{1j}, and X_{1j}' with X_{0j} (X_{1j}) $\stackrel{d}{=}$ X_{0j}' (X_{1j}') and define $\eta_{1j} = \eta_{2j} = X_{0j} + X_{1j}$, $\eta_{2j}' = X_{0j} + X_{1j}'$, and $\eta_{2j}'' = X_{0j}' + X_{1j}'$ $\forall\ j \geq 0$. We note that $\eta_{2j} \stackrel{d}{=} \eta_{2j}' \stackrel{d}{=} \eta_{2j}''$.

If we assume that X_{ij}, $X_{ij}' \sim G(\alpha_{ij}, b_{ij})$, $i = 0,1$, $j \geq 0$, then the sequence of EFFs { (η_{10}, η_{20}), (η_{11}, η_{21}), ... } will yield the LS model survival function (3.4), with $\alpha_{0j} + \alpha_{1j} = N_j\alpha$ and $b_{0j} = b_{1j} = b$ $\forall\ j \geq 0$. The CDF model (3.25) will result from the sequence of EFFs { (η_{10}, η_{20}'), (η_{11}, η_{21}'), ... } with $\alpha_{1j} = \alpha_{2j}$ and $b_{1j} = b_{2j}$ $\forall\ j \geq 0$. The sequence of EFFs { (η_{10}, η_{20}"), (η_{11}, η_{21}"), ... } gives an independent model with the following survival function:

$$\overline{F}_{\mathrm{IND}}(\tau_1,\tau_2) = \left[\frac{b_{n_1}}{\lambda_{1o}(\tau_1 - t_{n_1}) + b_{n_1}}\right]^{\alpha_{0n_1}+\alpha_{1n_1}} \left[\frac{b_{n_2}}{\lambda_{2o}(\tau_2 - t_{n_2}) + b_{n_2}}\right]^{\alpha_{0n_2}+\alpha_{1n_2}}$$

$$\cdot \left\{ \prod_{j=0}^{n_1-1} \left[\frac{b_j}{\lambda_{1o}(t_{j+1} - t_j) + b_j}\right]^{\alpha_{0j}+\alpha_{1j}} \prod_{j=0}^{n_2-1} \left[\frac{b_j}{\lambda_{2o}(t_{j+1} - t_j) + b_j}\right]^{\alpha_{0j}+\alpha_{1j}} \right\}, \tag{3.26}$$

where "IND" indicates that $\eta_{1j} \perp\!\!\!\perp \eta_{2j}''$, $j \geq 0$. η_{2j}, η_{2j}', and η_{2j}'' are identically distributed if we require that each a_{ij} and b_{ij} be equal across all three models (e.g., a_{00} (LS) $=$ a_{00} (CDF) $=$ a_{00} (IND)). Once this is done, it is clear that $(\eta_{1j}, \eta_{2j}) \overset{D}{>} (\eta_{1j}, \eta_{2j}') \overset{D}{>} (\eta_{1j}, \eta_{2j}'')$ $j = 0,1,\ldots$, and that the following theorem holds:

Theorem 3.6

$$\overline{F}_{IND}(\tau_1, \tau_2) < \overline{F}_{CDF}(\tau_1, \tau_2) < \overline{F}_{LS}(\tau_1, \tau_2)$$

for all τ_1 , $\tau_2 > 0$, and $\lambda_{1o}, \lambda_{2o} > 0$.

Thus the bivariate survival function of 2-component parallel redundant systems increases with the degree of dependence between dynamic piecewise continuous EFF's. Theorem 2.1 is a special case of 3.1, when $\tau_1, \tau_2 \in [\,0,\, t_1\,)$. This again generalizes for all m-component parallel redundant systems, $m \geq 2$.

CHAPTER IV

DEPENDENCE INDUCED BY DYNAMIC ENVIRONMENTS WITH ENVIRONMENTAL FACTOR FUNCTIONS DESCRIBED BY A CONTINUOUS TIME STOCHASTIC PROCESS

4.0 Introduction

In this chapter, we shall continue to assume that the operating environment is dynamic and that our uncertainty about the EFF $\eta(t)$ or its cumulative $Y(t) \equiv \int_0^t \eta(u)\, du$, is described by a continuous time stochastic process called the *gamma process*. The gamma process for the EFF will be motivated in section 4.1.1 as the limit of a piecewise constant EFF (see section 3.1.1) with independent gamma distributed innovations. The process, being nonnegative, nondecreasing in time, and possessing independent increments, is conceptually appealing. It has

been extensively studied; the first use has been attributed to Moran [1959]. In this dissertation, we rely primarily on results of Ferguson and Klass [1972], Cinlar [1980] and Dykstra & Laud [1981]; the use of gamma processes in survival analysis is primarily due to Ferguson [1973], Ferguson & Phadia [1979], and Kalbfleisch [1978].

Definition 4.1 Let $\alpha(t)$ be a nondecreasing left-continuous real valued function on $[0,\infty)$ with $\alpha(0) = 0$, and let $\beta \in (0, \infty)$. A stochastic process $\{ Y(t), t\geq 0 \}$ is said to be a *gamma process* with parameters $\alpha(t)$ and β, denoted "$Y(t) \in G_{pr}(\alpha(t), \beta)$", if:

1. $Y(0) = 0$,
2. $Y(t)$ has independent increments, and
3. $Y(t) - Y(s) \sim G(\alpha(t) - \alpha(s) , \frac{1}{\beta})$ for any $0 \leq s \leq t$.

The distribution $G(0, \beta)$ is defined as a distribution degenerate at 0.

Dykstra and Laud [1981] extend the gamma process to include a time-varying scale parameter $\beta(t)$.

Definition 4.2 Let $\beta(t)$, $t\geq 0$ be a positive right-continuous real valued function, and let $Y(t) \in G_{pr}(\alpha(t), 1)$. The process $Z(t) \equiv \int_0^t \beta(s)\, dY(s)$ is an *extended gamma process* denoted "$Z(t) \in G_{pr}(\alpha(t), \beta(t))$."

Properties of the gamma process (Cinlar [1980] ; Dykstra & Laud [1981]):

Let $Y(t) \in G_{pr}(\alpha(t), \beta)$. Then

$$E[\, Y(t) \,] = \int_0^t \beta \, d\alpha(u) \quad \text{and} \quad Var[\, Y(t) \,] = \int_0^t \beta^2 \, d\alpha(u) .$$

For any time t, $G^*_{Y(t)}$, the Laplace-Stieltjes transform (LST) of the distribution of Y(t), is

$$G^*_{Y(t)}(s) = \left[\frac{1}{1+\beta s} \right]^{\int_0^t d\alpha(u)}, \qquad s > 0 \text{ ; with}$$

$$G^*_{Y(t)}(s) = \left[\frac{1}{1+\beta s} \right]^{\alpha(t)}, \text{ if } \alpha(t) \text{ is continuously differentiable.}$$

Properties of the extended gamma process (Dykstra & Laud [1981]):

Let $Z(t) \in G_{pr}(\alpha(t), \beta(t))$. Then

$$E[\, Z(t) \,] = \int_0^t \beta(u) \, d\alpha(u) , \quad Var[\, Z(t) \,] = \int_0^t \beta^2(u) \, d\alpha(u), \text{ and}$$

$$G^*_{Z(t)}(s) = \exp\left[- \int_0^t \ln(\, 1 + s\beta(u) \,) \, d\alpha(u) \right], \qquad s > 0 ,$$

where $G^*_{Z(t)}$ is the LST of the distribution of Z(t).

In sections 4.1 and 4.3, we model the cumulative environmental factor function $Y(t) \equiv \int_0^t \eta(u) \, du$ as a gamma process; thus the cumulative failure rate $\Lambda(t) = \int_0^t \lambda_o(u) \, dY(u)$ is a gamma process if λ_o is constant, and an extended gamma process if λ_o varies with time. In section 4.2, we model the EFF itself as a gamma process.

4.1 Modeling the Cumulative EFF as a Gamma Process

We begin by assuming that the cumulative EFF at time τ is $Y(\tau) \in G_{pr}(\ \alpha(\tau), \frac{1}{b}\ )$, with $\alpha(\tau)$ continuously differentiable, and $\frac{d}{dt}\alpha(\tau) = a(\tau)$. The cumulative failure rate of component i, $\Lambda_i(\tau) = \int_0^\tau \lambda_{i_o}(u)\ dY(u),\ i = 1,2,$ is therefore an extended gamma process with parameters $(\ \alpha(\tau),\ \frac{\lambda_{i_o}(\tau)}{b}\ )$.

We motivate our choice of the gamma process for Y(t) in section 4.1.1. In section 4.1.2, we consider the special case of the baseline failure rate $\lambda_{i_o}(t)$ being a constant. This leads to several bivariate distributions depending upon the choice of shape parameter $\alpha(t)$; a special case is the bivariate exponential distribution of Marshall and Olkin [1967]. Thus we have a new derivation of this well known distribution. We continue by considering a linearly increasing baseline failure rate, which leads to a (possibly new) family of multivariate distributions with marginals that have a logarithmic failure rate function.

4.1.1 Motivation for Modeling the Cumulative EFF as a Gamma Process

In section 3.1.1, we assumed the environmental stress to be constant over small intervals $[0,t_1)$, $[t_1,t_2)$, … . We captured our knowledge about the stress over each interval $[t_j\ ,\ t_{j+1}\ )$, $j \geq 0$, via η_j. Our uncertainty about each η_j was described by a gamma distribution with

parameters ($N_j\alpha$, b), with η_j independent of η_i for all $j \neq i$. Thus the cumulative failure rate at any time τ, $0 \leq t_n < \tau \leq t_{n+1}$, for the ith component is:

$$\Lambda_i(\tau) = \lambda_{io}\left[\sum_{j=0}^{n-1} \eta_j \, (t_{j+1} - t_j) + \eta_n \, (\tau - t_n) \right], \quad i=1,2 \, .$$

For any finite positive time interval $\Delta t_j \equiv (t_{j+1} - t_j)$, it is possible to normalize the shape and scale parameters of the gamma distributions at each time t_j by defining

$$\alpha^*(t_j) \equiv \frac{N_j\alpha}{\Delta t_j} \quad \text{and} \quad b^* \equiv \frac{b}{\Delta t_j} \, .$$

Since we will be taking the limit as Δt_j tends to zero, we can assume that the intervals are uniform and that there exists an integer m such that $\tau = (m+1)\,\Delta t$. Thus the cumulative failure rate can be written as

$$\Lambda_i(\tau) = \lambda_{io}\left[\sum_{j=0}^{m} \eta_j \, \Delta t \right], \quad i=1,2.$$

Since $\eta_j \sim G(\alpha^*(t_j)\Delta t \, , \, b^*\Delta t)$, $\lambda_{io} \, \eta_j \, \Delta t \sim G(\alpha^*(t_j)\Delta t \, , \, \frac{b^*}{\lambda_{io}})$ and $\Lambda_i(\tau) \sim G(\sum_{j=0}^{m} \alpha^*(t_j)\Delta t \, , \frac{b^*}{\lambda_{io}})$, with characteristic function

$$\phi_{\Lambda_i(\tau)} = \left(1 - \frac{\sqrt{-1}\,\lambda_{io}}{b^*} \right)^{-\sum_{j=0}^{m} \alpha^*(t_j)\Delta t} .$$

As $\Delta t \downarrow 0$ ($m \uparrow \infty$), the characteristic function of $\phi_{\Lambda_i(\tau)}$ becomes:

$$\lim_{m\to\infty} \phi_{\Lambda_i(\tau)} = \left(1 - \frac{\sqrt{-1}\,\lambda_{io}}{b^*}\right)^{\lim_{m\to\infty} -\sum_{j=0}^{m} \alpha^*(t_j)\Delta t} = \left(1 - \frac{\sqrt{-1}\,\lambda_{io}}{b^*}\right)^{-\int_0^\tau \alpha^*(u)\,du} .$$

Thus for any $\tau \geq 0$, $\Lambda_i(\tau) \sim G(\int_0^\tau \alpha^*(u)\,du, \, \frac{b^*}{\lambda_{io}})$. If we consider any partition of $[\,0, \tau\,)$ such as $[\,0, s\,)$, $[\,s, \tau\,)$, then it can easily be shown that

$\Lambda_i(\tau) - \Lambda_i(s) \sim G(\int_s^\tau \alpha^*(u)\, du,\ \frac{b^*}{\lambda_{io}})$, with $\Lambda_i(\tau) - \Lambda_i(s)$ independent of $\Lambda_i(s)$, and $\Lambda_i(0) \sim G(0,\ \frac{b^*}{\lambda_{io}})$, implying that $\Lambda_i(0) = 0$. Thus we may conclude that the cumulative failure rate $\Lambda_i(\tau)$ of the piecewise constant independent innovations model of section 3.1.1 leads to a gamma process as the uniform time intervals $(t_{j+1} - t_j)$, $j \geq 0$, tend to zero.

4.1.2 Modeling the Cumulative EFF as a Gamma Process – The Bivariate Survival Function and its Marginals

Recall that describing $Y(\tau)$ by a gamma process with parameters $\alpha(\tau)$ and $\frac{1}{b}$ leads to an extended gamma process for $\Lambda_i(\tau)$ with parameters $\alpha(\tau)$ and $\frac{\lambda_{io}(\tau)}{b}$, $i = 1,2,\ldots,m$. For any continuously differentiable, increasing, and real valued function $\alpha(\tau)$ and any continuous positive real valued function $\frac{\lambda_{io}(\tau)}{b}$, with $\frac{d}{dt}\alpha(\tau) = a(\tau)$, the bivariate survival function for $0 \leq \tau_1 \leq \tau_2$ will be

$$\begin{aligned} \overline{F}(\tau_1, \tau_2) \equiv P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}(u), \lambda_{2o}(u), a(u), b;\ u \in [\, 0,\infty)\,] = \\ \exp\{ -\int_0^{\tau_1} \ln(1 + \frac{\lambda_{1o}(u)+\lambda_{2o}(u)}{b})\, a(u)\, du \} \\ \cdot \exp\{ -\int_{\tau_1}^{\tau_2} \ln(1 + \frac{\lambda_{2o}(u)}{b})\, a(u)\, du \}\ ; \end{aligned} \tag{4.1}$$

this is derived directly from the properties of the extended gamma process.

It is useful to examine the marginal distributions that arise from this formulation. The

marginal survival functions for $\tau_i \geq 0$ are

$$\overline{F}_i(\ \tau_i\ ) \equiv P[\ T_i \geq \tau_i \mid \lambda_{io}(u),\ a(u),\ b;\ u \in [\ 0,\infty\ )\ ] =$$

$$\exp\{\ -\int_0^{\tau_i} \ln(\ 1 + \frac{\lambda_{io}(u)}{b}\ )\ a(u)\ du\ \}\ , \qquad i=1,2\ . \tag{4.2}$$

Suppose that the baseline failure rate for component i, $\lambda_{io}(t) = \lambda_{io}$, a constant. Then the cumulative failure rate $\Lambda_i(\tau) = \lambda_{io} Y(\tau) \in G_{pr}(\ \alpha(\tau),\ \frac{\lambda_{io}}{b}\ )$, $i = 1,2$, thus for $0 \leq \tau_1 \leq \tau_2$,

$$\overline{F}(\ \tau_1,\ \tau_2\ ) \equiv P[\ T_1 \geq \tau_1,\ T_2 \geq \tau_2 \mid \lambda_{1o},\ \lambda_{2o},\ \alpha(u),\ b;\ u \in [\ 0,\ \infty\ )\ ] =$$

$$\left[\frac{b}{b + (\lambda_{1o} + \lambda_{2o})}\right]^{\alpha(\tau_1)} \left[\frac{b}{b + \lambda_{2o}}\right]^{\alpha(\tau_2)\ -\ \alpha(\tau_1)}, \tag{4.4}$$

with marginal survival and failure rate functions for $\tau_i \geq 0$

$$\overline{F}_i(\ \tau_i\ ) \equiv P[\ T_i \geq \tau_i \mid \lambda_{io},\ \alpha(u),\ b;\ u \in [\ 0,\infty\ )\ ] =$$

$$\left[\frac{b}{b + \lambda_{io}}\right]^{\alpha(\tau_i)}, \quad \text{and} \tag{4.5}$$

$$r_i(\tau_i) = a(\tau_i) \ln \left[\frac{b + \lambda_{io}}{b}\right], \quad i=1,2\ , \text{ respectively.} \tag{4.6}$$

The marginal failure rates consist of a constant times the derivative of the shape parameter $\alpha(\tau)$. By selecting different functional forms of $\alpha(\tau)$, each suggested by different physical models for the environment, we can derive different marginal distributions for the components. For example, if we describe $Y(t)$ using a gamma process, we might motivate it by describing the EFFs of the the piecewise constant innovations model of section 3.1.1 using iid gamma

variates. When we derive the limiting gamma process, the shape parameter $\alpha(t)$ will be linear.

If we suppose that the shape parameter is linear, say $\alpha(\tau) = \alpha_1 \tau$, $\alpha_1 > 0$, then the marginals are exponential, and the bivariate survival function for $0 \le \tau_1 \le \tau_2$ is

$$\bar{F}(\tau_1, \tau_2) \equiv P[T_1 \ge \tau_1, T_2 \ge \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \alpha_1, b]$$

$$= \left[\frac{b + \lambda_{2o}}{b + \lambda_{1o} + \lambda_{2o}}\right]^{\alpha_1 \tau_1} \left[\frac{b}{b + \lambda_{2o}}\right]^{\alpha_1 \tau_2}$$

$$= \left[\frac{b + \lambda_{2o}}{b + \lambda_{1o} + \lambda_{2o}}\right]^{\alpha_1 \tau_1} \left[\frac{b + \lambda_{1o}}{b + \lambda_{1o} + \lambda_{2o}}\right]^{\alpha_1 \tau_2} \left[\frac{b\,(b + \lambda_{1o} + \lambda_{2o})}{(b + \lambda_{2o})\,(b + \lambda_{1o})}\right]^{\alpha_1 \tau_2} . \tag{4.7}$$

If we let $\lambda_1^* \equiv \alpha_1 \ln\left[\dfrac{b + \lambda_{1o} + \lambda_{2o}}{b + \lambda_{2o}}\right]$, $\lambda_2^* \equiv \alpha_1 \ln\left[\dfrac{b + \lambda_{1o} + \lambda_{2o}}{b + \lambda_{1o}}\right]$

and $\lambda_{12}^* \equiv \alpha_1 \ln\left[\dfrac{(b + \lambda_{1o})\,(b + \lambda_{2o})}{b\,(b + \lambda_{1o} + \lambda_{2o})}\right]$, then

$$\bar{F}(\tau_1, \tau_2) = \exp - \left\{ \lambda_1^* \tau_1 + \lambda_2^* \tau_2 + \lambda_{12}^* \max[\tau_1, \tau_2] \right\}, \qquad \tau_1, \tau_2 \ge 0, \tag{4.8}$$

which is the bivariate exponential (BVE) distribution of Marshall and Olkin [1967], with marginals

$$\bar{F}_i(\tau_i) \equiv P[T_i \ge \tau_i \mid \lambda_{io}, \alpha_1, b] =$$

$$\exp - \left\{ (\lambda_i^* + \lambda_{12}^*)\,\tau_i \right\} = \exp - \left\{ \alpha_1 \ln\left[\frac{b + \lambda_{io}}{b}\right] \tau_i \right\},$$

$$\tau_i \ge 0, \quad i = 1, 2 . \tag{4.9}$$

It is interesting to note that under the appropriate assumptions, modeling the cumulative failure rate as a gamma process can lead to a standard shock model such as the BVE.

We can justify this development as follows. We know that the gamma process is a pure jump additive process - therefore if $\Lambda(t)$ is modeled as a gamma process, we can represent it as $\Lambda(t) = \sum_{j=1}^{\infty} X_j$, where X_j is a jump occurring at some time $T_j \in [\ 0,\ t\ )$. Although there are a countably infinite number of jumps occurring in $[\ 0,\ t\ )$, if we let $N(\ t,\ \epsilon\ )$ denote the number of jumps in $[\ 0,\ t\ )$ of magnitude $\geq \epsilon$ for any $\epsilon > 0$, then it can be shown (see, for example, Basawa and Prakasa Rao [1980], pp. 106-107) that $\{\ N(\ t,\ \epsilon\ ),\ t \geq 0\ \}$ is a Poisson process with an intensity function $m(\epsilon)$. If we consider increments of size Δx, then jumps of magnitude $X \geq k\Delta x$, $k = 1, 2, \ldots$, arrive in accordance with a Poisson process with intensity $m(k\Delta x)$, and we can proceed by decomposing the total arrival process to arrival processes of jumps of size $X \in [\ k\Delta x,\ (k+1)\Delta x\ )$, which will have intensities $\Delta m(\ k\Delta x\ ) \equiv m(\ (k+1)\Delta x\ ) - m(\ k\Delta x\ )$, $k = 1, 2, \ldots$.

The size of each jump is a random variable X distributed as $G_X(x)$. A jump in the cumulative failure rate represents a discrete shock to the system, which may cause the failure of one or both of the components. We expect that the probability that the shock causes the failure of a component to be a function of the size of the shock. If we choose an increment Δx sufficiently small, we can regard the probability of failure to be approximately the same for all $X \in [\ k\Delta x,\ (k+1)\Delta x\ )$. Let $p_{01}(k\Delta x)$ [$p_{10}(k\Delta x)$] be the probability that a shock of size $X \in [\ k\Delta x,\ (k+1)\Delta x\ )$ will cause the first (second) component to fail and the second (first) to

survive. Let $p_{00}(k\Delta x)$ [$p_{11}(k\Delta x)$] be the probability that a shock will cause both components to fail (survive). Thus we have for each arrival process of shocks of magnitude $X \in [\, k\Delta x, (k+1)\Delta x\,)$, $k = 1, 2, \ldots$, a standard nonfatal shock process with (only) a common shock applied to both components. Applying familiar techniques (see for example Mann, Schafer, and Singpurwalla [1974], pp. 148-149), we arrive at the following bivariate survival function for $\tau_1, \tau_2 \geq 0$:

$$P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid X \in [\, k\Delta x, (k+1)\Delta x\,)\,] =$$

$$\exp\{\, -\Delta m(k\Delta x)\, p_{01}(k\Delta x)\, \tau_1 \;-\Delta m(k\Delta x)\, p_{10}(k\Delta x)\, \tau_2 -\Delta m(k\Delta x)\, p_{00}(k\Delta x)\, \max(\tau_1,\tau_2)\,\}\,, \tag{4.10}$$

which represents the probability that the components will survive shocks of magnitude $X \in [\, k\Delta x, (k+1)\Delta x\,)$. The probability that the system will survive *all* shocks $\tilde{X}$ is the product of the incremental probabilities:

$$\begin{aligned}\bar{F}(\,\tau_1, \tau_2\,) &= P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid m(\epsilon), p_{01}(\epsilon), p_{10}(\epsilon), p_{00}(\epsilon); \epsilon > 0\,] = \\ &\exp\Big\{ -\sum_{k=0}^{\infty} \Delta m(k\Delta x)\, p_{01}(k\Delta x)\, \tau_1 \;-\sum_{k=0}^{\infty} \Delta m(k\Delta x)\, p_{10}(k\Delta x)\, \tau_2 \\ &\qquad -\sum_{k=0}^{\infty} \Delta m(k\Delta x)\, p_{00}(k\Delta x)\, \max(\tau_1,\tau_2)\Big\}\,. \end{aligned} \tag{4.11}$$

As we take the limit as $\Delta x \downarrow 0$, we have for $\tau_1, \tau_2 \geq 0$

$$\begin{aligned}\bar{F}(\,\tau_1, \tau_2\,) = \exp\Big\{ &-\int_0^{\infty} p_{01}(x)\, dm(x)\, \tau_1 \;-\int_0^{\infty} p_{10}(x)\, dm(x)\;\; \tau_2 \\ &-\int_0^{\infty} p_{00}(x)\, dm(x)\;\; \max(\tau_1,\tau_2)\Big\}\,. \end{aligned} \tag{4.12}$$

Thus the arrival rates in (4.8) are equivalent to the integral terms in (4.12).

To summarize, we see that a gamma process leads to the Marshall-Olkin bivariate exponential distribution derived from a non-fatal shock model with only a common shock applied to both components. The major difference between the derivation provided by Marshall and Olkin [1967] and this derivation lies in the fact that the original derivation assumed that all shocks have an identical probability of causing failure, while we assume that the shock (and thus the shock failure probability) is randomly distributed.

If we suppose that the gamma process shape parameter $\alpha(\tau)$ has the form $\alpha(\tau) = \alpha_p \, \tau^p$; p, $\alpha_p > 0$, then the marginal distributions are Weibull. The bivariate distribution for $0 \le \tau_1 \le \tau_2$ corresponding to marginal Weibulls is:

$$\bar{F}(\tau_1, \tau_2) \equiv P[\, T_1 \ge \tau_1, T_2 \ge \tau_2 \mid \lambda_{1o}, \lambda_{2o}, p, \alpha_p, b \,]$$

$$= \left[\frac{b + \lambda_{2o}}{b + \lambda_{1o} + \lambda_{2o}} \right]^{\alpha_p \tau_1^p} \left[\frac{b}{b + \lambda_{2o}} \right]^{\alpha_p \tau_2^p} . \tag{4.13}$$

If we define $\lambda_1^* \equiv \alpha_p \ln \left[\dfrac{b + \lambda_{1o} + \lambda_{2o}}{b + \lambda_{2o}} \right]$, $\quad \lambda_2^* \equiv \alpha_p \ln \left[\dfrac{b + \lambda_{1o} + \lambda_{2o}}{b + \lambda_{1o}} \right]$,

and $\lambda_{12}^* \equiv \alpha_p \ln \left[\dfrac{(b+\lambda_{1o})\,(b+\lambda_{2o})}{b\,(b+\lambda_{1o}+\lambda_{2o})} \right]$, then

$$\bar{F}(\tau_1, \tau_2) = \exp - \left\{ \lambda_1^* \tau_1^p + \lambda_2^* \tau_2^p + \lambda_{12}^* (\max[\tau_1, \tau_2])^p \right\},$$

$$\tau_1, \tau_2 \ge 0 , \tag{4.14}$$

which is a bivariate Weibull distribution, first identified by Marshall and Olkin [1967]. The marginal survival functions are

$\overline{F}_i(\tau_i) \equiv P[\, T_i \geq \tau_i \mid \lambda_{io}, p, \alpha_p, b\,] =$

$$\left[\frac{b}{b+\lambda_{io}}\right]^{\alpha_p \tau_i^p} = \exp - \left\{ \alpha_p \ln\left[\frac{b+\lambda_{io}}{b}\right] \tau_i^p \right\}, \qquad \tau_i \geq 0 . \tag{4.15}$$

Suppose that the components have a linearly increasing baseline failure rate, denoted by the differentiable function $\lambda_{io}(\tau) = b\,\beta_{i0} + b\,\beta_{i1} t$ for some b, β_{i0}, $\beta_{i1} > 0$, $i = 1,2$, and suppose that the cumulative EFF continues to be described ; y a gamma process with a linear, differentiable shape parameter function $\alpha(\tau) = \alpha_1 \tau$, $\tau \geq 0$. Then the cumulative failure rate $\Lambda_i(\tau) = \int_0^\tau \lambda_{io}(u)\, dYu$ forms an extended gamma process with shape function $\alpha(\tau)$ and scale function $\frac{\lambda_{io}(\tau)}{b}$. We use the LST of the increments of the extended gamma process to compute the survival distributions. For $0 \leq \tau_1 \leq \tau_2$:

$\overline{F}(\tau_1, \tau_2) \equiv P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \beta_{10}, \beta_{11}, \beta_{20}, \beta_{21}, \alpha_1, b\,] =$

$$\left[\frac{1+\beta_{10}+\beta_{20}}{1+\beta_{10}+\beta_{20}+(\beta_{11}+\beta_{21})\,\tau_1}\right]^{\frac{\alpha_1}{\beta_{11}+\beta_{21}}(1+\beta_{10}+\beta_{20})}$$

$$\cdot \left[\frac{1+\beta_{20}+\beta_{21}\tau_1}{1+\beta_{20}+\beta_{21}\tau_2}\right]^{\frac{\alpha_1}{\beta_{21}}(1+\beta_{20})} \cdot \left[\frac{1+\beta_{20}+\beta_{21}\tau_1}{1+\beta_{10}+\beta_{20}+(\beta_{11}+\beta_{21})\,\tau_1}\right]^{\alpha_1\tau_1}$$

$$\cdot \left[\frac{e}{1+\beta_{20}+\beta_{21}\,\tau_2}\right]^{\alpha_1\tau_2} . \tag{4.16}$$

This appears to be a *new* class of distributions (at least in a reliability context), with marginal survival functions for $\tau_i \geq 0$

$\overline{F}_i(\tau_i) \equiv P[\, T_i \geq \tau_i \mid \beta_{i0}, \beta_{i1}, \alpha_1, b\,] =$

$$\left[\frac{1+\beta_{i0}}{1+\beta_{i0}+\beta_{i1}\tau_i}\right]^{\frac{\alpha_1}{\beta_{i1}}(1+\beta_{i0})} \left[\frac{e}{1+\beta_{i0}+\beta_{i1}\tau_i}\right]^{\alpha_1\tau_i}, \tag{4.17}$$

and marginal failure rate functions for $\tau_i \geq 0$

$$r_i(\tau_i) = \alpha_1 \ln(1 + \beta_{i0} + \beta_{i1}\tau_i), \quad i=1,2, \tag{4.18}$$

which are logarithmic. The component marginal distributions are IFR, with the rate of increase of the failure rate steadily decreasing over time.

4.2 Modeling the EFF as a Gamma Process

In the previous section, we described the cumulative EFF $Y(\tau)$ using a gamma process. In this section, we describe the time-varying EFF $\eta(\tau)$ itself by a gamma process with parameters $(\alpha(\tau), b)$. This section is motivated by the work of Dykstra and Laud [1981], in which they model the component failure rate using a gamma process. The calculations involved in computing the characteristic functions of distributions with stochastic failure rates have been developed by Antelman and Savage [1965].

Although the gamma process has independent *increments*, it is easy to show that the EFF

$\eta(\tau)$ is autocorrelated in time. Thus Cov[$\eta(s)$, $\eta(t)$] = Var[$\eta(s)$].

The following theorem specifies the survival function of a single component system when the EFF is described by a gamma process.

Theorem 4.1. (Dykstra & Laud [1981]):

If the prior over hazard [failure] rates is $G_{pr}(\alpha(\cdot), \beta(\cdot))$, then the marginal survival function of an observation X is given by

$$\overline{F}(t) = \exp\left\{ - \int_0^t \ln(1 + \beta(u)\,(t-u))\, d\alpha(u) \right\}.$$

If arbitrary limits $0 \leq a \leq b$ are used in the proof of theorem 4.1, we get a result useful for deriving bivariate distributions.

Corollary 4.2.

Let $Z \in G_{pr}(\alpha(t), \beta(t))$. Then

$$E_Z[\exp\{ -s \int_a^b Z(u)\, du \}] = \exp\left\{ - \int_a^b \ln(1 + s\beta(u)\,(b-u))\, d\alpha(u) \right\}.$$

In section 4.2.1, we provide a motivation for using a gamma process to describe the EFF. In section 4.2.2, we present general results for bivariate and marginal distributions in cases where the baseline failure rate is allowed to vary over time. We then examine a special case where the gamma process shape parameter is linear and the baseline failure rate is constant, which provides the same marginal logarithmic failure rate functions as found in section 4.1.2.

4.2.1 Motivation for Modeling the EFF as a Gamma Process

In Section 3.1.2, we considered $\eta(t_j)$ as the sum of $j + 1$ independent gamma distributed random variables $c_0, \ldots, c_j$ with parameters α_j and b. It is possible to show that the sum $\eta(\tau)$ becomes a gamma process for any $\tau > 0$ in the limit as the time interval $[\, t_j, t_{j+1} \,)$ tends to zero. We assume that the intervals $[\, t_j, t_{j+1} \,) = \Delta t \;\; \forall\, j \geq 0$ and that there exists an integer m such that $\tau = (\, m+1 \,)\, \Delta t$.

We define for any finite positive Δt the parameters

$$\alpha^*(t_j) \equiv \frac{\alpha_j}{\Delta t} \quad \text{and} \quad b^* \equiv b\,.$$

Thus $c_j \sim G(\, \alpha^*(t_j)\Delta t\,,\, b^* \,)$ and $\eta(\tau) \sim G(\, \sum_{j=0}^{m} \alpha^*(t_j)\Delta t\,,\, b^* \,)$,

with characteristic function $\phi_{\eta(\tau)} = (\, 1 - \frac{\sqrt{-1}}{b^*} \,)^{-\sum_{j=0}^{m} \alpha^*(t_j)\Delta t}$.

As $\Delta t \downarrow 0$ (and $m \uparrow \infty$), the characteristic function of $\phi_{\eta(\tau)}$ becomes:

$$\lim_{m\to\infty} \phi_{\eta(\tau)} = (1 - \frac{\sqrt{-1}}{b^*})^{\lim_{m\to\infty} -\sum_{j=0}^{m} \alpha^*(t_j)\Delta t} = (1 - \frac{\sqrt{-1}}{b^*})^{-\int_0^{\tau} \alpha^*(u)\, du} .$$

Thus for any $\tau \geq 0$, $\eta(\tau) \sim G(\, \int_0^{\tau} \alpha^*(u)\, du,\; b^* \,)$. If we consider any partition of $[\, 0, \tau \,)$ such as $[\, 0, s \,)$, $[\, s, \tau \,)$, it can easily be shown that $\eta(\tau) - \eta(s) \sim G(\, \int_s^{\tau} \alpha^*(u)\, du,\; b^* \,)$, $\eta(\tau) - \eta(s)$ is independent of $\eta(s)$, and $\eta(0) = 0$.

From the above argument, we conclude that the EFF $\eta(\tau)$ of the cumulative piecewise gamma distribution modeled in section 3.1.2 becomes a gamma process in the limit as the time intervals ($t_{j+1} - t_j$) tend to zero.

4.2.2 Modeling the EFF as a Gamma Process — The Bivariate Survival Function and its Marginals

The EFF $\eta(t)$ continues to be described by a gamma process with parameters $\alpha(t)$ and $\frac{1}{b}$. If the baseline failure rate is a continuous, positive, real valued function of time, then we need the following theorem to derive the bivariate and marginal survival functions.

Theorem 4.3.

Let $\eta(t) \in G_{pr}(\alpha(t), \frac{1}{b})$, let $\lambda_o(t)$ be a known continuous positive real valued function, and let $\Lambda(t) = \int_0^t \lambda_o(u)\, \eta(u)\, du$. Then

$$\bar{F}(t \mid \lambda_o(t), \alpha(t), b;\ t \geq 0) = \exp\{ -\int_0^t \ln[1 + \frac{1}{b} \int_u^t \lambda_o(s)\, ds]\, d\alpha(u) \} .$$

Proof of Theorem 4.3:

Partition (0,t] into n x m parts. Let $h = \frac{t}{n}$. If $\eta(t) \in G_{pr}(\alpha(t), \frac{1}{b})$, then

$\eta(u) = \int_0^u \frac{1}{b}\, dX(s)$, where $X(t) \in G_{pr}(\alpha(t), 1)$. We can approximate the cumulative

failure rate $\Lambda(t)$ by a double sum; that is,

$$\Lambda(t) = \int_0^t \lambda_o(u)\left[\int_0^u \frac{1}{b}\, dX(s)\right] du \simeq \frac{1}{b}\sum_{j=1}^{n} \lambda_o(jh)\, h \sum_{i=1}^{jm}[\, X(\tfrac{ih}{m}) - X(\tfrac{(i-1)h}{m})\,] .$$

Let $Y_i = X(\frac{ih}{m}) - X(\frac{(i-1)h}{m})$. Then $Y_i \sim G(\, \alpha(\frac{ih}{m}) - \alpha(\frac{(i-1)h}{m}), 1\,)$, and

$\Lambda(t) \simeq \frac{h}{b}\sum_{j=1}^{n} \lambda_o(jh) \sum_{i=1}^{jm} Y_i$, and we can expand the right-hand side as

$$\frac{h}{b}\left\{ [\lambda_o(h)+\lambda_o(2h)+\cdots+\lambda_o(nh)] \sum_{i=1}^{m} Y_i + [\lambda_o(2h)+\cdots\lambda_o(nh)] \sum_{i=m+1}^{2m} Y_i + \cdots \right.$$

$$\left. + [\lambda_o(nh)] \sum_{i=(n-1)m+1}^{nm} Y_i \right\} .$$

We introduce for convenience a term W_j, where $W_j = \sum_{i=(j-1)m+1}^{jm} Y_i$, $j = 1,\ldots,n$. Then $W_j \sim G(\, \alpha(jh) - \alpha((j\text{-}1)h)\, ,\, 1\,)$ and $W_j \perp\!\!\!\perp W_k$, $j \neq k$, from the independence of increments of the original gamma process. We can now write $\Lambda(t)$ as

$$\Lambda(t) \simeq \frac{1}{b}\sum_{j=1}^{n} \left[\sum_{k=j}^{n} \lambda_o(kh)\, h\right] W_j .$$

Using this, we find the LST of $\Lambda(t)$.

$E[\exp\{-\Lambda(t)\}] \simeq \prod_{j=1}^{n} E[\exp\{-\frac{1}{b}[\sum_{k=j}^{n} \lambda_o(kh)\, h] W_j\}]$. Since

$W_j \sim G(\, \alpha(jh) - \alpha((j\text{-}1)h)\, , 1\,)$,

$E[\, e^{-s W_j}\,] = (\, 1 + s\,)^{-[\, \alpha(jh) - \alpha((j-1)h)\,]}$. Thus

$E[\, e^{-\Lambda(t)}\,] \simeq \prod_{j=1}^{n} [\, 1 + \frac{1}{b} \sum_{k=j}^{n} \lambda_o(kh)\, h\,]^{-[\, \alpha(jh) - \alpha((j-1)h)\,]}$. To get an exact value for $E[\, e^{-\Lambda(t)}\,]$, we take the limit of the approximating sums as $n \uparrow \infty$ (thus $h \downarrow 0$). Therefore

$$E[\, e^{-\Lambda(t)}\,] = \lim_{n\to\infty} \exp\{ - \sum_{j=1}^{n} [\, \alpha(jh) - \alpha((j\text{-}1)h)\,] \ln[\, 1 + \frac{1}{b} \sum_{k=j}^{n} \lambda_o(kh)\, h\,] \},$$

and the survival function is

$$\overline{F}(\, t\,) = E[\, e^{-\Lambda(t)}\,] = \exp\{ - \int_0^t \ln[\, 1 + \frac{1}{b}\int_u^t \lambda_o(s)\, ds\,]\, d\alpha(u) \} . \qquad \square$$

Note that if $\lambda_o(t) = \lambda_o \ \forall \ t \geq 0$ (constant), then Theorem 4.3 is equivalent to Theorem 4.1.

The bivariate survival function for $0 \leq \tau_1 \leq \tau_2$ follows directly from Theorem 4.3. Thus

$\overline{F}(\tau_1, \tau_2) \equiv P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}(u), \lambda_{2o}(u), a(u), b;\ u \in [\,0,\infty\,)\,] =$

$$\overline{F}(\tau_1,\tau_2) = \exp\{ -\int_0^{\tau_1} \ln[\, 1 + \frac{1}{b}\int_u^{\tau_1} (\lambda_{1o}(s)+\lambda_{2o}(s))\, ds \,]\, a(u)\, du \,\}$$

$$\cdot \exp\{ -\int_{\tau_1}^{\tau_2} \ln[\, 1 + \frac{1}{b}\int_u^{\tau_2} \lambda_{2o}(s)\, ds \,]\, a(u)\, du \,\} \,, \tag{4.19}$$

with marginal survival functions for $\tau_i \geq 0$

$\overline{F}(\tau_i) \equiv P[\, T_i \geq \tau_i \mid \lambda_{io}(u), a(u), b;\ u \in [\,0,\infty\,)\,] =$

$$\exp\{ -\int_0^{\tau_i} \ln[\, 1 + \frac{1}{b}\int_u^{\tau_i} \lambda_{io}(s)\, ds \,]\, a(u)\, du \,\} \,,$$

and failure rate functions for $\tau_i \geq 0$ (4.20)

$$r_i(\tau_i) = \frac{\lambda_{io}(\tau_i)}{b} \int_0^{\tau_i} [\, 1 + \frac{1}{b}\int_u^{\tau_i} \lambda_{io}(s)\, ds \,]^{-1} a(u)\, du \,, \quad i=1,2 \,. \tag{4.21}$$

Once again, we can choose plausible functional forms for $\alpha(t)$ and $\lambda_{io}(t)$, suggested by the physical model of the environment, that allow for a closed-form solution for the distribution functions. Choices of $\alpha(t)$ and $\lambda_{io}(t)$ that lead to interesting marginal distributions are provided below, along with their respective bivariate distributions.

One special case of interest in our motivation of the gamma process (section 4.2.1) is the process as a limit of the sum of iid gamma variates. If c_j is identically distributed for all j, $j = 0,1,\ldots$, with $\eta(t) = \sum_{j \mid t_j < t} c_j$, then $\eta(t) - \eta(s) = \sum_{j \mid s \leq t_j < t} c_j$ will be described by a gamma

distribution with a shape parameter that depends on the length of the interval (t-s), which will lead in the limit to a gamma process with a linear shape function $\alpha(t)$, say $\alpha(t) = \alpha_1 t$, for some $\alpha_1 > 0$, $t \geq 0$.

We assume that $\eta(\tau) \in G_{pr}(\ \alpha_1\tau,\ \frac{1}{b}\ )$, which implies that the component failure rate is $\lambda_i(\tau) = \lambda_{io}\ \eta(\tau) \in G_{pr}(\ \alpha_1\tau,\ \frac{\lambda_{io}}{b}\ )$. For convenience in notation, we let $\beta_{11} \equiv \frac{\lambda_{1o}}{b}$ and $\beta_{21} \equiv \frac{\lambda_{2o}}{b}$.

The bivariate survival function for $0 \leq \tau_1 \leq \tau_2$ can be found using Corollary 4.2.

$$\overline{F}(\ \tau_1, \tau_2\ ) \equiv P[\ T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \beta_{11}, \beta_{21}, \alpha_1\ ] =$$

$$= (\ 1 + (\beta_{11}+\beta_{21})\tau_1\ )^{-\frac{\alpha_1}{(\beta_{11}+\beta_{21})}}\ (\ 1 + \beta_{21}(\ \tau_2-\tau_1\ )\ )^{-\frac{\alpha_1}{\beta_{21}}}$$

$$\cdot \left[\frac{1 + \beta_{21}(\ \tau_2-\tau_1\ )}{1 + (\beta_{11}+\beta_{21})\ \tau_1}\right]^{\alpha_1\tau_1} \cdot \left[\frac{e}{1 + \beta_{21}(\ \tau_2-\tau_1\ )}\right]^{\alpha_1\tau_2}. \tag{4.22}$$

The marginal survival functions for $\tau_i \geq 0$ are

$$\overline{F}(\ \tau_i\ ) \equiv P[\ T_i \geq \tau_i \mid \beta_{i1}, \alpha_1\ ]$$

$$= (\ 1 + \beta_{i1}\tau_i\ )^{-\frac{\alpha_1}{\beta_{i1}}} \left[\frac{e}{1 + \beta_{i1}\ \tau_i}\right]^{\alpha_1\tau_i}, \tag{4.23}$$

with failure rate functions for $\tau_i \geq 0$

$$r_i(\tau_i) = \alpha_1 \ln[\, 1 + \beta_{i1}\tau_i \,] \ , \ i=1,2 \, . \tag{4.24}$$

It is interesting to note that the marginal distributions are the same as those obtained in the previous section using an extended gamma process with shape parameter $\alpha(\tau) = \alpha_1\tau$ and scale parameter $\frac{\lambda_{io}(\tau)}{b} = \beta_{i1}\,\tau$. However, we have a different bivariate distribution leading to this type of marginal distribution.

If we suppose that the component baseline times-to-failure are described by independent Weibull distributions, then $\lambda_{io}(t) = p\ \lambda_{io}^p\ t_i^{p-1} \quad p>0 \quad i=1,2$. The marginal failure rate functions have closed form expressions if the shape parameter $\alpha(t)$ of the EFF gamma process is a $p-$degree function of time. If we suppose that $\alpha(t) = \alpha_p\, t^p$, $\alpha_p > 0$, then $a(t) = \frac{d}{dt}\alpha(t)$ $= p\ \alpha_p\ t^{p-1}$, and the failure rate is

$$r_i(t) = p\ \alpha_p\ t^{p-1} \ln[\, 1 + \frac{\lambda_{io}^p}{b}\, t^p \,] \qquad p,\ \alpha_p > 0\ ;\ t \geq 0,\ i=1,2\, . \tag{4.25}$$

For example, if we let $p = 1$, then $\alpha(t) = \alpha_1 t$, $a(t) = \alpha_1$, and $\lambda_{io}^p\ t^p = \lambda_{io}\ t$. Thus

$$r_i(t) = \alpha_1 \ln[\, 1 + \frac{\lambda_{io}}{b}\, t \,] \ , \qquad i=1,2 \ ,$$

which is consistent with the results obtained previously. Bivariate distributions can easily be derived using Theorem 4.3.

4.3 Modeling the Cumulative EFF as a Gamma Process Having Different Effects on Each Component

In this section, we use the approach embodied in the Cherian-David-Fix class of bivariate distributions to define a dependent set of gamma processes, assuming that some environmental stress elements affect the components in the same manner, and other stress elements affect the components differently. We describe the cumulative effect of the common stress at time τ with a gamma process denoted as $Y_0(\tau)$; that is, $Y_0(\tau) \in G_{pr}(\ \alpha_0(t),\ \frac{1}{b}\ )$. We describe the cumulative effect of the stress unique to component i at time τ with a gamma process denoted as $Y_i(\tau)$; that is, $Y_i(\tau) \in G_{pr}(\ \alpha_i,\ \frac{1}{b}\ )$, $i = 1,2$. We also assume that the processes have mutually independent increments; thus for any $0 \le s \le t$ and any $0 \le u \le v$, $Y_j(t)-Y_j(s) \perp\!\!\!\perp Y_i(v)-Y_i(u)$, $j \neq i$, $i,j = 0,1,2$. The component baseline failure rates are assumed constant, so $\Lambda_i(t) \equiv \int_0^t \lambda_{io}(u)\ \eta_i(u)\ du = \lambda_{io}\ [\ Y_0(t) + Y_i(t)\ ]$, $i = 1,2$. This model yields the bivariate survival function for $0 \le \tau_1 \le \tau_2$ as

$$\overline{F}(\ \tau_1, \tau_2\ ) \equiv P[\ T_1 \ge \tau_1, T_2 \ge \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \alpha_i(u), b;\ u \in [0,\infty),\ i=0,1,2\ ]$$

$$= \left[\frac{b}{b + \lambda_{1o} + \lambda_{2o}}\right]^{\alpha_0(\tau_1)} \left[\frac{b}{b + \lambda_{2o}}\right]^{\alpha_0(\tau_2)-\alpha_0(\tau_1)} \left[\frac{b}{b + \lambda_{1o}}\right]^{\alpha_1(\tau_1)}$$

$$\cdot \left[\frac{b}{b + \lambda_{2o}}\right]^{\alpha_2(\tau_2)}, \qquad (4.26)$$

with marginal survival functions for $\tau_i \geq 0$

$$\bar{F}(\ \tau_i\ ) \equiv P[\ T_i \geq \tau_i \mid \lambda_{io},\ \alpha_l(u),\ b;\ u \in [0,\infty),\ l=0,i\ ] =$$

$$= \left[\frac{b}{b+\lambda_{io}}\right]^{\alpha_0(\tau_i)+\alpha_i(\tau_i)}, \tag{4.27}$$

and marginal failure rate functions

$$r_i(\tau_i) = (\ a_0(\tau_i) + a_i(\tau_i)\ )\ \ln\left[\frac{b+\lambda_{io}}{b}\right], \quad i=1,2\ . \tag{4.28}$$

The marginal results obtained here are essentially the same as those obtained from Section 4.1.2. The marginal distributions may be IFR, DFR, or have mixed increasing / decreasing failure rates, depending upon the shape functions $\alpha_i(\tau)$, $i = 0,1,2$, chosen. For example, if the processes $Y_i(\tau)$ have a linear shape function $\alpha_i(\tau)$, the marginal distributions are exponential, and if the shape functions are of the form $\alpha_{ip}\ t^p$, $\alpha_{ip} > 0$, $p > 0$, then the marginal distributions are Weibull.

4.4 Inequalities for Survival Functions of Continuous EFFs With Increasing Degrees of Dependence

The bivariate survival functions can be compared under increasing degrees of dependence for the continuous distributions of sections 4.1 and 4.3. Let $Y_0(\tau)$, $Y_0'(\tau)$, $Y_1(\tau)$, $Y_1'(\tau)$ denote gamma processes, with $Y_0(\tau) \triangleq Y_0'(\tau)$, $Y_1(\tau) \triangleq Y_1'(\tau)$, and $Y_i(\tau)$, $Y_i'(\tau)$ mutually

independent, $i = 0,1$, $\forall\ \tau \geq 0$. We assume $Y_i(t) \in G_{pr}(\ \alpha_i(t), \frac{1}{b}\ )$ and we construct three pairs of random functions $(\ \Lambda_1(t), \Lambda_2(t)\ )$, $(\ \Lambda_1(t), \Lambda_2'(t)\ )$, and $(\ \Lambda_1(t), \Lambda_2''(t)\ )$ by defining $\frac{\Lambda_1(t)}{\lambda_{1o}} = \frac{\Lambda_2(t)}{\lambda_{2o}} = Y_0(t) + Y_1(t)$, $\frac{\Lambda_2'(t)}{\lambda_{2o}} = Y_0(t) + Y_1'(t)$, and $\frac{\Lambda_2''(t)}{\lambda_{2o}} = Y_0'(t) + Y_1'(t)$. In this section, it is more convenient to work with the cumulative failure rate functions $\Lambda(t)$ than the EFFs $\eta(t)$, noting that the cumulative EFF for component i is $\int_0^t \eta_i(u)\,du = \frac{\Lambda_i(t)}{\lambda_{io}}$, $i = 1,2$.

The LS model, where $\frac{\Lambda_1(t)}{\lambda_{1o}} = \frac{\Lambda_2(t)}{\lambda_{2o}}$, yields the survival function for $0 \leq \tau_1 \leq \tau_2$

$$\bar{F}_{LS}(\ \tau_1,\tau_2\ ) \equiv P[\ T_1 \geq \tau_1,\ T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \alpha_0(u), \alpha_1(u), b;\ u \in [0,\infty)\ ]$$

$$= \left[\frac{b}{b + \lambda_{1o} + \lambda_{2o}}\right]^{\alpha_0(\tau_1)+\alpha_1(\tau_1)}$$

$$\cdot \left[\frac{b}{b + \lambda_{2o}}\right]^{[\alpha_0(\tau_2)+\alpha_1(\tau_2)]-[\alpha_0(\tau_1)+\alpha_1(\tau_1)]} . \tag{4.29}$$

The Cherian-David-Fix (CDF) model using $\Lambda_1(t)$, $\Lambda_2'(t)$ yields for $0 \leq \tau_1 \leq \tau_2$

$$\bar{F}_{CDF}(\ \tau_1,\tau_2\ ) \equiv P[\ T_1 \geq \tau_1,\ T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \alpha_0(u), \alpha_1(u), b;\ u \in [0,\infty)\ ] =$$

$$\left[\frac{b}{b + \lambda_{1o} + \lambda_{2o}}\right]^{\alpha_0(\tau_1)} \left[\frac{b}{b + \lambda_{2o}}\right]^{[\alpha_0(\tau_2)+\alpha_1(\tau_2)]-\alpha_0(\tau_1)} \left[\frac{b}{b + \lambda_{1o}}\right]^{\alpha_1(\tau_1)} , \tag{4.30}$$

and the independent model with $\Lambda_1(t)$, $\Lambda_2''(t)$ gives for $0 \leq \tau_1 \leq \tau_2$

$$\bar{F}_{IND}(\ \tau_1,\tau_2\ ) \equiv P[\ T_1 \geq \tau_1,\ T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \alpha_0(u), \alpha_1(u), b;\ u \in [0,\infty)\ ]$$

$$= \left[\frac{b}{b + \lambda_{1o}}\right]^{\alpha_0(\tau_1)+\alpha_1(\tau_1)} \left[\frac{b}{b + \lambda_{2o}}\right]^{\alpha_0(\tau_2)+\alpha_1(\tau_2)} . \tag{4.31}$$

It is easy to verify that theorem 4.4 holds.

Theorem 4.4

Let $\overline{F}_{IND}(\tau_1,\tau_2)$, $\overline{F}_{CDF}(\tau_1,\tau_2)$, and $\overline{F}_{LS}(\tau_1,\tau_2)$ be defined as above. Then

$$\overline{F}_{IND}(\tau_1,\tau_2) < \overline{F}_{CDF}(\tau_1,\tau_2) < \overline{F}_{LS}(\tau_1,\tau_2)$$

for τ_1 , $\tau_2 > 0$ and $\lambda_{1o}, \lambda_{2o} > 0$.

Thus the bivariate survival function of 2-component parallel redundent systems increases with the degree of dependence between cumulative EFFs described by identically distributed gamma processes. This result extends to *m*-component systems and is expected based on the results of section 3.4 and the limiting arguments in section 4.1.1.

APPENDIX A

NOTATION

α, $\alpha(t)$	Shape parameter of a gamma distribution or process.
α_i, $\alpha_i(t)$, $i=1,\ldots,m$	Shape parameter of a gamma distribution or process, referring to component i.
α_j, $j=0,1,\ldots$	Shape parameter of a gamma distribution or process, referring to the EFF η_j.
a, a(t)	Derivative of the shape parameter of a gamma distribution or process.
b	Scale parameter of a gamma distribution or process.
β, β_k, $k=0,1,\ldots,s$	Unknown coefficient or known distribution parameter.
CDF	Refers to models for EFFs $\eta_1(t)$ and $\eta_2(t)$ that use the Cherian-David-Fix bivariate density (see Chapter 2).
c_k, $k=0,1,\ldots,s$	Factor used in section 3.1 to describe the effect of the kth stress on the system.

$\eta(t)$	Environmental Factor function (EFF) common to all components
η_j, $j=0,1,\ldots$	EFF over time interval $[\, t_j, t_{j+1})$, common to both components (see Chapter 3).
$\eta_i(t)$, $i=1,\ldots,m$	EFF for component i.
G, G_j, G_k, $j=0,1,\ldots$, $k=0,1,\ldots,s$	A distribution function for a random variable.
G(α, b)	Gamma distribution with density $$dG(x) = \frac{b^{\alpha}\ x^{\alpha-1}\ e^{-bx}}{\Gamma(\alpha)}, \qquad x > 0.$$
$G_{pr}(\alpha(t), \beta)$	Gamma process with parameters $\alpha(t)$ and β.
$G_{pr}(\alpha(t), \beta(t))$	Extended gamma process with parameters $\alpha(t)$ and $\beta(t)$.
h(t)	Attenuation function for the shot-noise model of section 3.2.
H(t)	$H(t) = \int_0^t h(u)\, du.$
i, $i = 0,1,\ldots,m$	Subscript referring to component i.
$I_k(t_j)$, $k=0,1,\ldots,s$, $j=0,1,\ldots$	Indicator variable used in section 3.1. $I_k(t_j) =$ 1 if kth stress is present at time t_j, 0 otherwise.
IND	Refers to independent models for EFFs $\eta_1(t)$ and $\eta_2(t)$.

j, $j = 0,1,\ldots$	Subscript referring to time, related to interval $[\,t_j,\, t_{j+1}\,)$.
k, $k = 0,1,\ldots,s$	Subscript referring to a stress k or a factor c_k used to describe the effect of stress k.
λ_{io}, $\lambda_{io}(t)$, $i=1,2,\ldots,m$	Baseline failure rate for component i.
λ_i, $\lambda_i(t)$, $i=1,2,\ldots,m$	Total failure rate for component i. $\lambda_i(t) = \lambda_{io}(t)\, \eta_i(t)$
$\Lambda(t)$	Cumulative failure rate for a single component; $\Lambda(t) = \int_0^t \lambda_o(u)\, \eta(u)\, du$.
$\Lambda_i(t)$, $i=1,\ldots,m$	Cumulative failure rate for component i.
λ_1^*, λ_2^*, λ_{12}^*	Failure rate parameters of the Marshall-Olkin bivariate exponential distribution.
LS	Refers to model derived in Lindley & Singpurwalla [1986].
m, m(t)	Intensity function for a Poisson process.
M(t)	$M(t) = \int_0^t m(u)\, du$.
n_1, n_2	Indices related to times τ_1 and τ_2, such that $0 \le t_{n_1} \le \tau_1 < t_{n_1+1} \le t_{n_2} \le \tau_2 < t_{n_2+1}$.
N_j, $j=0,1,\ldots$	Number of stresses present during interval $[\,t_j,\, t_{j+1}\,)$.

p_{kj}, q_{kj}; $k=0,1,\ldots,s$, $j=0,1,\ldots$	p_{kj} is the probability that the kth stress is present during time interval $[\, t_j,\, t_{j+1}\,)$. $q_{kj} = 1 - p_{kj}$.
$\underline{t} = (\, t_1, t_2, \ldots\,)$	Vector of (known) transition times for the piecewise continuous model of Chapter 3.
$[\, t_j,\, t_{j+1}\,)$, $j=0,1,\ldots$	Time interval over which the EFF is constant in the models used in Chapter 3.
T_i, $i=1,\ldots,m$	Lifelength of component i.
τ	Time at which a univariate distribution is evaluated; $\tau \geq 0$.
τ_1, τ_2	Times at which a bivariate or marginal distribution is evaluated; τ_1, $\tau_2 \geq 0$.
Δt	$\Delta t = [\, t_j,\, t_{j+1}\,)\ \forall\ j \geq 0$ if time intervals are uniform.
$Y(t)$	Cumulative EFF; $Y(t) = \int_0^t \eta(u)\, du$.
$Y_i(t)$ $i=1,\ldots,m$	Cumulative EFF for component i.

APPENDIX B

DETERMINISTIC EFFs WITH UNKNOWN COEFFICIENTS

In the introduction, we stated that a routine extention of the LS model is to describe the dynamic environment using an environmental factor function $\eta(\tau)$ that is deterministic with unknown coefficients. A polynomial function of time is used to demonstrate general results.

Suppose that the effect of a dynamic environment on the system over a finite range $[\ 0,\ T'\ )$ can be captured through a polynomial environental factor function $\eta(\tau)$. We suppose that the degree of the polynomial, p, is known or assumed but the coefficient values are unknown. We denote the coefficients as β_k. $k = 0,1,\ldots,p$, denote the vector of coefficients as $\underline{\beta} = (\ \beta_0,\ \beta_1,\ \ldots,\ \beta_p\ )$, and describe our uncertainty about each β_k with a distribution G_k. We further assume that the β_k's are mutually independent $\forall\ k$, $k = 0,1,\ldots,p$. Thus the EFF $\eta(\tau) = \sum_{k=0}^{p} \beta_k\, \tau^k,\ \ p \geq 0,\ \tau \in [\ 0,\ T'\ )$.

Since we restricted the range of τ to an interval $[\ 0,\ T'\ )$, we can allow negative coefficients; thus the EFF $\eta(t)$ may increase or decrease within the interval $[\ 0,\ T'\ )$. The support of the β_k's may be nonnegative or restricted such that $\eta(\tau) > 0\ \forall\ \tau \in [\ 0,\ T'\ )$.

B-1. A Polynomial EFF with Gamma Distributed Coefficients

Suppose that our uncertainty about each of our coefficients β_k can be described using a gamma distribution with known parameters α_k and b_k. Then our system survival function will be for any $\tau \in [\, 0, T')$

$\bar{F}(\, \tau \,) \equiv P[\, T \geq \tau \mid \lambda_{1o}, \lambda_{2o}, \alpha_k, b_k, k{=}0,1,\ldots,s \,] =$

$$\prod_{k=0}^{p} \left[\frac{(k+1)b_k}{\lambda_{1o}\, \tau^{k+1} + (k+1)b_k} \right]^{\alpha_k} + \prod_{k=0}^{p} \left[\frac{(k+1)b_k}{\lambda_{2o}\, \tau^{k+1} + (k+1)b_k} \right]^{\alpha_k}$$

$$- \prod_{k=0}^{p} \left[\frac{(k+1)b_k}{(\lambda_{1o}+\lambda_{2o})\, \tau^{k+1} + (k+1)b_k} \right]^{\alpha_k},$$

with a bivariate survival function for $\tau_1, \tau_2 \in [\, 0, T')$

$\bar{F}(\, \tau_1, \tau_2 \,) \equiv P[\, T_1 \geq \tau_1, T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \alpha_k, b_k, k{=}0,1,\ldots,s \,]$

$$= \prod_{k=0}^{p} \left[\frac{(k+1)b_k}{\lambda_{1o}\tau_1^{k+1} + \lambda_{2o}\tau_2^{k+1} + (k+1)b_k} \right]^{\alpha_k}.$$

Our marginal survival functions will be for $\tau_i \in [\, 0, T')$

$\bar{F}_i(\, \tau_i \,) \equiv P[\, T_i \geq \tau_i \mid \lambda_{io}, \alpha_k, b_k, k{=}0,1,\ldots,s \,]$

$$= \prod_{k=0}^{p} \left[\frac{(k+1)b_k}{\lambda_{io}\tau_i^{k+1} + (k+1)b_k} \right]^{\alpha_k}, \quad i{=}1,2.$$

B-2. A Polynomial EFF with Coefficients Distributed as a Truncated Normal

Suppose that our uncertainty about each of our coefficients β_k can be described using a truncated normal distribution with known parameters μ_k and σ_k^2, truncated from below at some value β_k^*. We denote this distribution as $TN_{\beta_k^*}(\mu_k, \sigma_k^2)$. Our density function for each β_k will be

$$
\begin{aligned}
dG_k(\beta_k) &= \frac{1}{c_k\sqrt{2\pi\sigma_k^2}} \exp\left[-\frac{1}{2\sigma_k^2}(\beta_k - \mu_k)^2\right], & \beta_k > \beta_k^*, \\
&= 0, & \beta_k \leq \beta_k^*,
\end{aligned}
$$

$$
\text{where} \qquad c_k \equiv \int_{\beta_k^*}^{\infty} \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left[-\frac{1}{2\sigma_k^2}(x - \mu_k)^2\right] dx .
$$

We need to choose our lower truncation point β_k^* of the distribution G_k in such a way as to guarantee that $\eta(\tau) > 0 \ \forall \ \tau \in [0, T')$. To do this, we establish the following proposition:

Proposition B-1: Define t^* such that $\sum_{k=0}^{p} \beta_k^*(t^*)^k = \min_{t \in [0,T']} \sum_{k=0}^{p} \beta_k^* t^k$. If $\sum_{k=0}^{p} \beta_k^*(t^*)^k > 0$, then $\eta(t) > 0 \ \forall \ t > 0$.

Proof of Proposition B-1: $\eta(t) = \sum_{k=0}^{p} \beta_k t^k \geq \sum_{k=0}^{p} \beta_k^* t^k \geq \sum_{k=0}^{p} \beta_k^*(t^*)^k > 0$ □

Thus, to insure that $\eta(\tau) > 0$, we can adjust the values β_k^* by trial and error until $\sum_{k=0}^{p} \beta_k^*(t^*)^k > 0$. Note that β_o^* must be positive since $\eta(0) > 0$. We provide several examples below on how to find each β_k^*. Obviously, this procedure will become tedious for large values of p unless we set most, if not all, $\beta_k^* \geq 0$.

Example 1: p = 1 (linear function)

We know that $\beta_o^* > 0$. If $\beta_1^* \geq 0$, then $\sum_{k=0}^{p} \beta_k^* \tau^k > 0 \;\; \forall \; \tau \geq 0$. Suppose $\beta_1^* < 0$. By definition, $\beta_o^* + \beta_1^* t^* = \min_{t \in [0,T']} \beta_o^* + \beta_1^* t$. A linear function is minimized at either endpoint; if $\beta_1^* < 0$, then the function is decreasing and its minimum is reached at T'. Thus the condition is met for any $\beta_1^* \geq \frac{-\beta_o^*}{T'}$.

Example 2: p = 2 (quadratic function)

Again, we know that $\beta_o^* > 0$. If β_1^* and $\beta_2^* \geq 0$, then $\sum_{k=0}^{p} \beta_k^*(t^*)^k > 0$. Suppose that $\beta_1^* < 0$ and $\beta_2^* < 0$. Then $\beta_o^* + \beta_1^* t + \beta_2^* t^2$ is minimized at t = T'. Therefore, one can select any β_1^* and β_2^* such that $\beta_1^* + \beta_2^* T' \geq \frac{-\beta_o^*}{T'}$. Now suppose that $\beta_1^* < 0$ and $\beta_2^* > 0$. A quadratic will be minimized at the endpoints or at a point where the first derivative equals 0. Thus $t^* = 0$, T', or $-\beta_1^*/2\beta_2^*$. We know that if $t^* = 0$, then $\sum_{k=0}^{p} \beta_k^*(t^*)^k = \beta_o^* > 0$. Any value of β_1^* and β_2^* such that $\sum_{k=0}^{p} \beta_k^*(t^*)^k > 0$ for both $t^* = T'$ and $t^* = -\beta_1^*/2\beta_2^*$ will be satisfactory. The latter condition also holds if we suppose that $\beta_1^* > 0$ and $\beta_2^* < 0$.

After we have determined the parameters β_k^* for every k, $k = 0,1,\ldots,p$, then we can

easily calculate the system survival function for any $\tau \geq 0$:

$$\overline{F}(\tau) \equiv P[\, T \geq \tau \mid \lambda_{1o}, \lambda_{2o}, \mu_k, \sigma_k^2, \beta_k^*, {\scriptstyle k=0,1,\ldots,s} \,] =$$

$$\prod_{k=0}^{p} G_k^*\left(\frac{\lambda_{1o}\,\tau^{k+1}}{k+1} \right) + \prod_{k=0}^{p} G_k^*\left(\frac{\lambda_{2o}\,\tau^{k+1}}{k+1} \right) + \prod_{k=0}^{p} G_k^*\left(\frac{(\lambda_{1o}+\lambda_{2o})\,\tau^{k+1}}{k+1} \right)$$

$$= \prod_{k=0}^{p} K_k\left(\frac{\lambda_{1o}\,\tau^{k+1}}{k+1} \right) \exp\left[-\mu_k \frac{\lambda_{1o}\,\tau^{k+1}}{k+1} + \frac{\sigma_k^2 \left(\frac{\lambda_{1o}\,\tau^{k+1}}{k+1}\right)^2}{2} \right]$$

$$+ \prod_{k=0}^{p} K_k\left(\frac{\lambda_{2o}\,\tau^{k+1}}{k+1} \right) \exp\left[-\mu_k \frac{\lambda_{2o}\,\tau^{k+1}}{k+1} + \frac{\sigma_k^2 \left(\frac{\lambda_{2o}\,\tau^{k+1}}{k+1}\right)^2}{2} \right]$$

$$- \prod_{k=0}^{p} K_k\left(\frac{(\lambda_{1o}+\lambda_{2o})\,\tau^{k+1}}{k+1} \right)$$

$$\cdot \exp\left[-\mu_k \frac{(\lambda_{1o}+\lambda_{2o})\,\tau^{k+1}}{k+1} + \frac{\sigma_k^2 \left(\frac{(\lambda_{1o}+\lambda_{2o})\,\tau^{k+1}}{k+1}\right)^2}{2} \right].$$

If we define Φ as the standard normal cdf, then

$$c_k = 1 - \Phi\left[\frac{\beta_k^* - \mu_k}{\sigma_k}\right], \qquad a_k(s) = 1 - \Phi\left[\frac{\beta_k^* - (\mu_k - \sigma_k^2 s)}{\sigma_k}\right], \text{ and}$$

$$K_k(s) \equiv \frac{a_k(s)}{c_k} = \frac{1 - \Phi\left[\frac{\beta_k^* - (\mu_k - \sigma_k^2 s)}{\sigma_k}\right]}{1 - \Phi\left[\frac{\beta_k^* - \mu_k}{\sigma_k}\right]}.$$

The bivariate survival function is, for $0 \leq \tau_1 \leq \tau_2 < T'$:

$$\overline{F}(\tau_1, \tau_2) \equiv P[\, T_1 \geq \tau_1,\ T_2 \geq \tau_2 \mid \lambda_{1o}, \lambda_{2o}, \mu_k, \sigma_k^2, \beta_k^*, {\scriptstyle k=0,1,\ldots,s}\,] =$$

$$\prod_{k=0}^{p} \left\{ K_k\!\left(\frac{\lambda_{1o}\, \tau_1^{k+1} + \lambda_{2o}\, \tau_2^{k+1}}{k+1} \right) \right.$$

$$\left. \cdot \exp\left[-\mu_k \frac{\lambda_{1o}\, \tau_1^{k+1} + \lambda_{2o}\, \tau_2^{k+1}}{k+1} + \frac{\sigma_k^2 \left(\frac{\lambda_{1o}\, \tau_1^{k+1} + \lambda_{2o}\, \tau_2^{k+1}}{k+1} \right)^2}{2} \right] \right\},$$

with marginal survival functions for $\tau_i \in [\, 0, T')$

$$\overline{F}_i(\tau_i) \equiv P[\, T_i \geq \tau_i \mid \lambda_{io}, \mu_k, \sigma_k^2, \beta_k^*, {\scriptstyle k=0,1,\ldots,s}\,] =$$

$$\prod_{k=0}^{p} K_k\!\left(\frac{\lambda_{io}\, \tau_i^{k+1}}{k+1} \right) \exp\left[-\mu_k \frac{\lambda_{io}\, \tau_i^{k+1}}{k+1} + \frac{\sigma_k^2 \left(\frac{\lambda_{io}\, \tau_i^{k+1}}{k+1} \right)^2}{2} \right].$$

REFERENCES

Antelman, G. and Savage, I.R. (1965). Characteristic functions of stochastic integrals and reliability theory. *Naval Research Logistics Quarterly*, Vol. 12, pp. 199-222.

Barlow, R.E. (1985). A Bayes' explanation of an apparent failure rate paradox. *IEEE Transactions in Reliability*, Vol. R-34, No. 2, pp. 107-108.

Basawa, I.V. and Prakasa Rao, B.L.S. (1980) *Statistical Inference for Stochastic Processes.* Academic Press, London.

Cherian, K.C. (1941). A bivariate correlated gamma type distribution function. *Journal of the Indian Mathematical Society*, Vol. 5, pp. 133-144.

Çinlar, E. (1980). On a generalization of gamma processes. *Journal of Applied Probability*, Vol. 17, pp. 467-480.

Çinlar, E. and Özekici, S. (1987). Reliability of Complex Devices in Random Environments. *Probability in the Engineering and Informational Sciences*, Vol. 1, pp. 97-115.

Cox, D. R. (1972). Regression models and life tables (with discussion). *Journal of the Royal Statistical Society Series B*, Vol. 34, pp. 187-220.

Cox, D. R. and Isham, V. (1980). *Point Processes*. Chapman and Hall, London.

Currit, A. and Singpurwalla, N.D. (1987). On the reliability function of a system of components sharing a common environment. *Journal of Applied Probability*. To appear.

David, F.N. and Fix, E. (1961). Rank correlation and regressions in a non-normal surface. *Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, J. Neyman, ed., University of California Press, pp. 177-197.

Downton, F. (1970). Bivariate exponential distributions in reliability theory. *Journal of the Royal Statistical Society Series B*, Vol. 32, pp. 408-417.

Dykstra, R.L. and Laud, P. (1981). A Bayesian nonparametric approach to reliability. *The Annals of Statistics*, Vol. 9, No. 2, pp. 356-367.

Ferguson, T.S. (1973). A Bayesian analysis of some nonparametric problems. *The Annals of Statistics*, Vol. 1, No. 2, pp. 209-230.

Ferguson, T.S. and Klass, M.J. (1972). A representation of independent increment processes without Gaussian components. *The Annals of Mathematical Statistics*, Vol. 43, No. 5, pp. 1634-1643.

Ferguson, T.S. and Phadia, E.G. (1979). Bayesian nonparametric estimation based on censored data. *The Annals of Statistics*, Vol. 7, No. 1, pp. 163-186.

Flournoy, N. (1982). The failure-censoring bichain and the relative efficiency of selected partial liklihoods in the presence of co-processes. Ph.D. dissertation, University of Washington.

Freund, J.E. (1961). A bivariate extension of the exponential distribution. *Journal of the American Statistical Association*, Vol. 56, pp. 971-977.

Gamerman, D. and West, M. (1987). Dynamic survival models in action. Unpublished technical report, University of Warwick.

Goel, P.K. and DeGroot, M. (1979). Bayesian estimation and optimal designs in partially accelerated life testing. *Naval Research Logistics Quarterly*, Vol. 26, pp. 223-235.

Johnson, N.L. and Kotz, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions.* John Wiley and Sons, Inc., New York.

Kalbfleisch, J.D. (1978). Nonparametric Bayesian analysis of survival time data. *Journal of the Royal Statistical Society Series B*, Vol. 40, No. 2, pp. 214-221.

Kalbfleisch, J.D. and Prentice, R.L. (1980). *The Statistical Analysis of Failure Time Data.* John Wiley and Sons, Inc., New York.

Lemoine, A.J. and Wenocur, M.L. (1985). On failure modeling. *Naval Research Logistics Quarterly*, Vol. 32, pp. 497-508.

Lemoine, A.J. and Wenocur, M.L. (1986). A note on shot noise and reliability modeling. *Operations Research*, Vol. 34, No. 2, pp. 320-323.

Lindley, D.V. and Singpurwalla, N.D. (1986). Multivariate distributions for the life lengths of c[illegible]nents of a system sharing a common environment. *Journal of Applied Probability*, Vol. 23, No. 2, pp. 418-431.

Malik, H.J. and Abraham, B. (1973). Multivariate logistic distributions. *Annals of Statistics*, Vol. 3, pp. 588-590.

Mann, N.R., Schafer, R.E., and Singpurwalla, N.D. (1974). *Methods for Statistical Analysis of Reliability and Life Data.* John Wiley & Sons, Inc., New York.

Marshall, A.W. and Olkin, I. (1967). A multivariate exponential distribution. *Journal of the American Statistical Association*, Vol. 62, pp. 30-44.

Moran, P.A.P. (1959). *The Theory of Storage.* Methuen, London.

Nayak, T.K. (1987). Multivariate Lomax distribution: properties and usefulness in reliability theory. *Journal of Applied Probability*, No. 24, pp. 170-177.

Shaked, M. and Singpurwalla, N.D. (1983). Inference for step-stress accelerated life tests. *Journal of Statistical Planning and Inference*, Vol. 7, pp. 295-306.

Singpurwalla, N.D. (1988). Foundational issues in reliability and risk analysis. *SIAM Review*. To appear.